The National **Numeracy** *Strategy*

Teaching mental calculation strategies

guidance for teachers at key stages 1 and 2

QCA

Guarding standards

The Qualifications and Curriculum Authority would like to thank all the consultants who put their efforts and expert knowledge into producing this guidance. We are also grateful to the many teachers, local education authorities, subject associations and other organisations who gave us valuable advice and feedback during its development. Advice was also sought from members of the Numeracy Task Force, the National Numeracy Strategy, Ofsted and the Teacher Training Agency.

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Introduction

The ability to calculate 'in your head' is an important part of mathematics and an important part of coping with society's demands and managing everyday events. The national curriculum and the *Framework for teaching mathematics* make clear that children should learn number facts by heart and be taught to develop a range of mental strategies for quickly finding from known facts a range of related facts that they cannot recall rapidly. There are several ways of carrying out calculations: using paper and pencil methods, using a calculator, working them out mentally or a combination of these. Many children in key stages 1 and 2 do not use the most efficient methods to carry out calculations. For much work at key stages 1 and 2, a mental approach to calculations is often the most efficient and needs to be taught explicitly.

Informal recording and the use of tools such as number lines and hundred number squares can be used to develop understanding of number and help to develop competence and confidence for mental calculation at all stages. Calculators can help to develop a better sense of number. However, this guidance makes it clear that children should not use calculators for calculations until they can at least add and subtract any pair of two-digit numbers in their head. Thus, regular use of a calculator should not begin until year 5, when this expectation is usually first met.

The purpose of this booklet

The purpose of this booklet is to offer guidance to teachers on teaching effective mental strategies for calculation and to make clear the expectations concerning the use of calculators. It is designed to assist teachers in their planning by:

- listing those number facts that children are expected to recall rapidly;
- giving clear year-by-year expectations of a range of calculations children should be able to do mentally;
- listing those strategies that might be introduced to children for them to learn and practise as an aid to performing calculations;
- suggesting a range of suitable activities for use in the classroom.

Part 1

Part 1 describes what calculation facts children are expected to recall rapidly and what sorts of calculations they might be able to perform mentally. Both are set out side-by-side on a year-by-year basis. The expectations set out in this booklet are the same as those set out in the *Framework for teaching mathematics*. The booklet presupposes that recall knowledge from an earlier year is always carried through into subsequent years. The section also includes a list of strategies for mental calculation that are appropriate for year-by-year development. Such a list cannot be exhaustive; the strategies chosen are widely considered to be the key strategies children need to develop their mental skills and understanding of number.

This document on mental strategies should be used in conjunction with the *Framework for teaching mathematics* and the QCA publication *Standards in mathematics* and so help schools to plan and teach the primary mathematics curriculum.

Part 2

Part 2 discusses choosing the most appropriate strategies for a range of calculations of varying degrees of difficulty. Discussing with children the relative merits of different strategies helps them to see why some strategies are more appropriate and efficient than others. Talking through a strategy will aid children's learning and sharpen their ability to perform calculations mentally. Key questions for teachers to ask include: 'How did you work that out?', 'Who did it another way?', 'Which is the easier/easiest way?'.

What should be stressed is the idea of the efficiency of a method, by which is meant the ease and speed with which it leads to a correct answer. Different strategies may often be used to help perform a particular calculation, but some may be more efficient than others. Children can be encouraged to discuss the relative merits of a range of strategies applied to a range of computations. The aim is that children choose a sensible strategy for a calculation. However, many children are unable to retain all the information in their head when performing mental calculations. For such children, informal recording should be encouraged. Children should also be asked to explain their mental calculation strategies and to record the results of a mental calculation in a number sentence, using the correct signs and symbols.

Part 2 ends with a short section that describes how a teacher might incorporate different aspects of the teaching and learning of mental mathematics into the typical three-part daily mathematics lesson recommended by the National Numeracy Strategy for implementation in schools from September 1999.

Parts 3 and 4

Parts 3 and 4 specify the key strategies to teach children and contain activities to support the teaching of these strategies. Part 3 deals with addition and subtraction and Part 4 with multiplication and division. Each subsection begins with examples of typical problems addressed; these are set out on a year-by-year basis and match expectations described in Part 1.

Children need to be given sufficient time to master each strategy so that they not only use it but also can explain why it is useful.

The National Numeracy Strategy places great emphasis on whole class teaching. The activities to support the teaching and learning of the mathematics and mental calculation strategies described in the different subsections should, wherever possible, be used with the whole class to engage all the children in the same task. When group work is undertaken, some activities will prove useful to groups of children who need further consolidation in order to progress, and some will help to extend the thinking of children who need a greater degree of challenge.

The activities are indicative of the sort of work that can help children to learn and practise a strategy or to consolidate knowledge and understanding prior to mastering a particular strategy.

Part 5

Part 5 discusses the appropriate use of calculators.

It is important to realise that calculators should not replace mental skills. They are not appropriate for computations that can quickly be done mentally. Calculators, however, should be used in cases where neither a mental method nor a written method is the most appropriate.

Children have access to a range of calculating aids and tools inside and out of the classroom. Teachers must carefully consider when to use which aid and tool. Calculators can be very effective teaching tools, for example to show pattern in number situations such as multiplying by 10, and in reinforcing concepts in place value, such as that 367 is 300 + 60 + 7.

Too early a use of calculators can foster dependency. However, using calculators to help develop an understanding of number operations and the structure of number will help children to become more fluent with number. There are also constructive ways to use calculators to explore and teach aspects of the structure of the number system. It is most likely to be in years 5 and 6 that problems become sufficiently complicated for most children to warrant the use of the calculator as a calculating aid.

The statutory national tests at key stage 2 now include a mental arithmetic test as well as pencil and paper calculations to be undertaken without a calculator and a test where a calculator can be used.

Part 6

Part 6 of this booklet deals with approximating and checking. It draws together ideas from all the previous sections to show how children consolidate their learning and understanding of the mathematical principles set out in this booklet. Work in this area improves children's confidence and facility in tackling numerical problems and calculations.

Glossary

A glossary of terms used in the main pages of this booklet is also included.

Formal written methods

This guidance is about teaching mental strategies to children. However, writing mathematics is also a significant feature of learning about the subject. There are several aspects to 'writing' mathematics. These include learning to use mathematical notation such as the correct use of the 'equals' sign (=) or the 'greater than' (>) or 'less than' (<) signs, or writing expressions involving operations (eg 5 x 10 = 50), or using brackets. Developing appropriate mathematical terminology (in reading or in written work) is also essential. Formal written methods are also needed.

Formal written methods should be taught after children have a firm grounding in a range of mental strategies. The *Framework for teaching mathematics* recommends that formal written methods should be introduced from year 4.

Part 1
Mental calculation: expectations for each year

It is important that teachers of all years should address all the strategies, including the basic strategies, to ensure that they are well covered. This section reinforces the expectations set out by the *Framework for teaching mathematics*. Teachers should refer to the 'Framework' for further examples of mental calculations.

	Rapid recall **Children should be able to recall rapidly:**	Mental strategies **Children should be able to use the following strategies, as appropriate, for mental calculations:**
Year 1	– all pairs of numbers with a total of 10, eg 3 + 7; – addition and subtraction facts for all numbers to at least 5; – addition doubles of all numbers to at least 5, eg 4 + 4.	– count on or back in ones; – reorder numbers in a calculation; – begin to bridge through 10, and later 20, when adding a single-digit number; – use known number facts and place value to add or subtract pairs of single-digit numbers; – add 9 to single-digit numbers by adding 10 then subtracting 1; – identify near doubles, using doubles already known; – use patterns of similar calculations.
Year 2	– addition and subtraction facts for all numbers to at least 10; – all pairs of numbers with a total of 20, eg 13 + 7; – all pairs of multiples of 10 with a total of 100, eg 30 + 70; – multiplication facts for the 2 and 10 times-tables and corresponding division facts; – doubles of all numbers to ten and the corresponding halves; – multiplication facts up to 5 x 5, eg 4 x 3.	– count on or back in tens or ones; – find a small difference by counting up from the smaller to the larger number; – reorder numbers in a calculation; – add three small numbers by putting the largest number first and/or find a pair totalling 10; – partition additions into tens and units then recombine; – bridge through 10 or 20; – use known number facts and place value to add or subtract pairs of numbers; – partition into '5 and a bit' when adding 6, 7, 8 or 9, then recombine; – add or subtract 9, 19, 11 or 21 by rounding and compensating; – identify near doubles; – use patterns of similar calculations; – use the relationship between addition and subtraction; – use knowledge of number facts and place value to multiply or divide by 2, 5 or 10; – use doubles and halves and halving as the inverse of doubling.

Mental calculations

Children should be able to calculate mentally:

- add or subtract a single-digit to or from a single-digit, without crossing 10, eg 4 + 5, 8 – 3;
- add or subtract a single-digit to or from 10;
- add or subtract a single-digit to or from a 'teens' number, without crossing 20 or 10, eg 13 + 5, 17 – 3;
- doubles of all numbers to 10, eg 8 + 8, double 6.

- add or subtract any single-digit to or from any two-digit number, without crossing the tens boundary, eg 62 + 4, 38 – 7;
- add or subtract any single-digit to or from a multiple of 10, eg 60 + 5, 80 – 7;
- add or subtract any 'teens' number to any two-digit number, without crossing the tens boundary, eg 23 + 14, 48 – 13;
- find what must be added to any two-digit multiple of 10 to make 100, eg 70 + ? = 100;
- add or subtract a multiple of 10 to or from any two-digit number, without crossing 100, eg, 47 + 30, 82 – 50;
- subtract any two-digit number from any two-digit number when the difference is less than 10, eg 78 – 71, or 52 – 48;
- doubles of all numbers to at least 15, eg double 14;
- double any multiple of 5 up to 50, eg double 35;
- halve any multiple of 10 up to 100, eg halve 50.

	Rapid recall	Mental strategies
	Children should be able to recall rapidly:	**Children should be able to use the following strategies, as appropriate, for mental calculations:**
Year 3	– addition and subtraction facts for all numbers to 20; – all pairs of multiples of 100 with a total of 1000; – all pairs of multiples of 5 with a total of 100; – multiplication facts for the 2, 5 and 10 times-tables and corresponding division facts.	– count on or back in tens or ones; – find a small difference by counting up from the smaller to the larger number; – reorder numbers in a calculation; – add three or four small numbers by putting the largest number first and/or by finding pairs totalling 9, 10 or 11; – partition into tens and units then recombine; – bridge through a multiple of 10, then adjust; – use knowledge of number facts and place value to add or subtract pairs of numbers; – partition into '5 and a bit' when adding 6, 7, 8 or 9; – add or subtract mentally a 'near multiple of 10' to or from a two-digit number; – identify near doubles; – use patterns of similar calculations; – say or write a subtraction statement corresponding to a given addition statement; – to multiply a number by 10/100, shift its digits one/two places to the left; – use knowledge of number facts and place value to multiply or divide by 2, 5, 10, 100; – use doubling or halving; – say or write a division statement corresponding to a given multiplication statement.
Year 4	– multiplication facts for 2, 3, 4, 5 and 10 times-tables; – division facts corresponding to tables of 2, 3, 4, 5 and 10.	– count on or back in repeated steps of 1, 10 and 100; – count up through the next multiple of 10, 100 or 1000; – reorder numbers in a calculation; – add 3 or 4 small numbers, finding pairs totalling 10; – add three two-digit multiples of 10; – partition into tens and units, adding the tens first; – bridge through 100; – use knowledge of number facts and place value to add or subtract any pair of two-digit numbers; – add or subtract 9, 19, 29, 11, 21 or 31 by rounding and compensating; – add or subtract the nearest multiple of 10 then adjust; – identify near doubles; – continue to use the relationship between addition and subtraction; – double any two-digit number by doubling tens first; – use known number facts and place value to multiply or divide, including multiplying and dividing by 10 and then 100; – partition to carry out multiplication; – use doubling or halving; – use closely related facts to carry out multiplication and division; – use the relationship between multiplication and division.

Mental calculations

Children should be able to calculate mentally:

- find what must be added to any multiple of 100 to make 1000, eg 300 + ? = 1000;
- add or subtract any pair of two-digit numbers, without crossing a tens boundary or 100, eg 33 + 45, 87 – 2;
- add or subtract any single-digit to any two-digit number, including crossing the tens boundary, eg 67 + 5, 82 – 7;
- find what must be added to/subtracted from any two-digit number to make the next higher/lower multiple of 10, eg 64 + ? = 70, 56 – ? = 50;
- subtract any three-digit number from any three-digit number when the difference is less than 10, eg 458 – 451, or 603 – 597;
- find what must be added to/subtracted from any three-digit number to make the next higher/lower multiple of 10, eg 647 + ? = 650, 246 – ? = 240;
- doubles:
  - ▾ double any number to at least 20, eg double 18, and corresponding halves, eg halve 36;
  - ▾ double 60, halve 120;
  - ▾ double 35, halve 70;
  - ▾ double 450, halve 900;
- multiply single-digit numbers by 10 or 100, eg 6 x 100;
- divide any multiple of 10 by 10, eg 60 ÷ 10, and any multiple of 100 by 100, eg 700 ÷ 100.

- find what must be added to any two-digit number to make 100, eg 37 + ? = 100;
- add or subtract any pair of two-digit numbers, eg 38 + 85, 92 – 47;
- find out what must be added to/subtracted from any two- or three-digit number to make the next higher/lower multiple of 100, eg 374 + ? = 400, 826 – ? = 800;
- subtract any four-digit number from any four-digit number when the difference is small, eg 3641 – 3628, 6002 – 5991;
- doubles and halves:
  - ▾ double any whole number from 1 to 50, eg double 36, and find all the corresponding halves, eg 96 ÷ 2;
  - ▾ double any multiple of 10 to 500, eg 380 x 2, and find all the corresponding halves, eg 760 ÷ 2, 130 ÷ 2;
  - ▾ double any multiple of 5 to 100, eg 65 x 2;
- multiply any two-digit number by 10, eg 26 x 10;
- divide a multiple of 100 by 10, eg 600 ÷ 10;
- multiply any two-digit multiple of 10 by 2, 3, 4 or 5, eg 60 x 4, 80 x 3.

	Rapid recall	Mental strategies
	Children should be able to recall rapidly:	**Children should be able to use the following strategies, as appropriate, for mental calculations:**
Year 5	– multiplication facts to 10 x 10; – division facts corresponding to tables up to 10 x 10.	– count up through the next multiple of 10, 100 or 1000; – reorder numbers in a calculation; – partition into hundreds, tens and units, adding the most significant digit first; – use known number facts and place value to add or subtract pairs of three-digit multiples of 10 and two-digit numbers with one decimal place; – add or subtract the nearest multiple of 10 or 100 then adjust; – identify near doubles; – add several numbers; – develop further the relationship between addition and subtraction; – use factors; – partition to carry out multiplication; – use doubling and halving; – use closely related facts to carry out multiplication and division; – use the relationship between multiplication and division; – use knowledge of number facts and place value to multiply or divide.
Year 6	– squares of all integers from 1 to 10.	– consolidate all strategies from previous years; – use knowledge of number facts and place value to add or subtract pairs of three-digit multiples of 10 and two-digit numbers with one decimal place; – add or subtract the nearest multiple of 10, 100 or 1000, then adjust; – continue to use the relationship between addition and subtraction; – use factors; – partition to carry out multiplication; – use doubling and halving; – use closely related facts to carry out multiplication and division; – use the relationship between multiplication and division; – use knowledge of number facts and place value to multiply or divide.

Mental calculations

Children should be able to calculate mentally:

- add or subtract any pair of three-digit multiples of 10, eg 570 + 250, 620 – 380;
- find what must be added to a decimal fraction with units and tenths to make the next higher whole number, eg. 4.3 + ? = 5;
- add or subtract any pair of decimal fractions each with units and tenths, or each with tenths and hundredths, eg 5.7 + 2.5, 0.63 – 0.48;
- subtract a four-digit number just less than a multiple of 1000 from a four-digit number just more than a multiple of 1000, eg 5001 – 1997;
- multiply any two- or three-digit number by 10 or 100, eg 79 x 100, 363 x 100;
- divide a multiple of 100 by 10 or 100, eg 4000 ÷ 10, 3600 ÷ 100;
- multiply any two-digit multiple of 10 by a single-digit, eg 60 x 7, 90 x 6;
- double any whole number from 1 to 100, multiples of 10 to 1000, and find corresponding halves;
- find 50%, 25%, 10% of a small whole numbers or quantities, eg 25% of £8.

- multiply any two-digit number by a single-digit, eg 34 x 6;
- multiply any two-digit number by 50 or 25, eg 23 x 50, 47 x 25;
- multiply or divide any whole number by 10 or 100, giving any remainder as a decimal, eg 47 ÷ 10 = 4.7, 1763 ÷ 100 = 17.63;
- find squares of multiples of 10 to 100;
- find any multiple of 10% of a whole number or quantity, eg 70% of £20, 50% of 5kg, 20% of 2 metres.

Part 2
Teaching strategies for mental calculation

Is mental calculation the same as mental arithmetic?

For many people, mental arithmetic is mainly about rapid recall of number facts – knowing your addition bonds to 20 and the tables to 10 x 10.

This rapid recall of number facts is one aspect of mental calculation. However, there is another that involves challenging children with calculations where they have to figure out the answer rather than recall it from a bank of number facts that are committed to memory.

Examples:

A year 2 child who knows that 9 + 9 = 18 can use this to calculate mentally other results, eg 9 + 8, 9 + 18, 19 + 9, 19 + 19.

A year 4 child who knows that 6 x 4 = 24 can use this fact to calculate 12 x 4 by doubling.

A year 6 child who knows that 36 ÷ 4 = 9 can use this to calculate 56 ÷ 4 by partitioning 56 into 36 + 20 and using the knowledge of 36 ÷ 4 and 20 ÷ 4 to reach the answer of 14.

The tables in Part 1 indicate what the *Framework for teaching mathematics* expects children to know with rapid recall and to be able to calculate mentally in each year of key stage 1 and key stage 2.

Research shows that committing some number facts to memory helps children develop strategies, and that the use of strategies to figure out answers helps them commit further facts to memory. It is therefore important that schools focus on teaching mental strategies for calculation.

What's special about mental calculations?

Carrying out a calculation mentally is not the same as doing a traditional paper and pencil algorithm while mentally picturing it in your head rather than putting it on paper.

Example:

Children in a year 4 class are carrying out mental addition of two-digit numbers.

Jo explains that 36 + 35 must be 71 (double 35 plus 1) (near doubles).

Sam explains that 36 + 45 is 36 plus 40 (76) plus 4 (80) plus 1, giving 81 (partitioning the smaller number and then bridging to 10).

Misha explains that 38 + 37 is 60 (30 plus 30) plus 15 (known fact: 8 + 7 = 15), giving 75.

These explanations demonstrate two key features of mental calculation:

- numbers are treated in a holistic way, as quantities rather than as digits;
- calculations that appear similar can be amenable to the use of different strategies depending on the numbers involved.

These two features are in contrast to carrying out the calculations using a standard written method where the numbers are treated as a collection of digits to be manipulated, and a set procedure is followed irrespective of the numbers involved or the order of difficulty. For example, all the following calculations would be treated the same way if the standard pencil and paper algorithm for subtraction were used: 61 – 4, 61 – 41, 61 – 32, 61 – 58, 61 – 43. If carried out mentally, the numbers will provoke the use of different strategies.

It is clear that there are advantages in expecting children to use mental calculation methods for calculations which might traditionally have been done with a standard paper and pencil algorithm. Such advantages include a stronger 'number sense', better understanding of place value and more confidence with numbers and the number system.

How do I help children develop a range of mental strategies?

Individual children will be at different stages in terms of the number facts that they have committed to memory and the strategies available to them for figuring out other facts. This booklet has been written to help develop awareness of the possible range of strategies that can be taught. Being aware of the range of strategies has these advantages for the teacher:

- when children are carrying out mental calculations the teacher may be better able to recognise the strategies being used;
- the teacher can draw attention to and model a variety of strategies used by children in the class;
- suggestions can be made to the children that will move them on to more efficient strategies.

There are three aspects to developing a range of mental strategies and ensuring that children become effective in deploying these strategies:

- raising children's awareness that there is a range of strategies;
- working on children's confidence and fluency with a range of strategies;
- developing efficient methods.

(i) Raising children's awareness that there is a range of strategies

Examples:

Some eight-year-olds are working on this problem: three cars, three bicycles and two lorries go past the school gate? How many wheels went by?

After agreeing that there are eight wheels on each lorry, Tom and Sam quickly agree that there are 12 + 6 + 16 wheels. Tom counts on from 12 and announces 33 as his answer. Sam, after a few moments' reflection, announces that it is 34. Asked to explain his method, he replies: 'Well there's 12 and the 10 from there [pointing to the 16] makes 22, there's another six left [from the 16], so that's 28. Two from there [the 6] makes 30 and there's four left so that's 34.'

A reception teacher is working with a 0 to 10 number line and getting the children to count on using it.

She deliberately chooses examples of the sort 1 + 6, 1 + 8 and so on to provoke the children into putting the larger number first. Sure enough, one of the children suggests this, and the teacher picks up the strategy and works with the children explicitly on it. Afterwards, she explains that had no one suggested this after a couple of examples she would have pointed it out to the children herself.

There are several advantages in using realistic contexts to provoke calculation, including:

- if the context is sufficiently interesting, most children are motivated to find a solution;
- the context is likely to suggest methods to finding a solution, and these methods can be discussed.

(ii) Working on children's confidence and fluency with a range of strategies

The bulk of this booklet provides guidance on teaching methods for particular strategies. Two general methods that support the development of strategies and enable children to become increasingly competent and efficient are:

- working on related calculations. For many children, their experience of working on mathematical calculations results in their coming to treat each calculation as a new one. Careful selection of problems can help provoke different strategies. For example, having established that 32 – 12 = 20, the children can be asked to provide answers for 32 – 13, 31 – 12, 42 – 12, and so on.
- talking through methods of solution. Asking children to explain their methods of solution has the advantages that:
  - the child doing the explaining clarifies the thinking;
  - the children listening are introduced to the idea that there are a variety of methods;
  - some methods may be more efficient than ones they are currently using.

Classroom talk focuses on the structure of number and why certain methods work, and so supports children's conceptual development and their powers of mathematical reasoning.

(iii) Developing efficient methods

Individual children will be at different stages in terms of their understanding of the number system and the number facts they have committed to memory. These features will have a bearing on the strategies available to them for figuring out other facts and performing mental calculations.

Example:

A year 2 class is working on multiples of 4.

Tara has committed to memory doubles up to double 10. She readily uses this knowledge to figure out the first few multiples of 4 by doubling twice to give 4, 8, 12, 16, 20.

Shane has not yet committed his doubles to memory, but has learned the pattern of counting in fours. He rapidly uses this pattern to multiply by 4, up to 20.

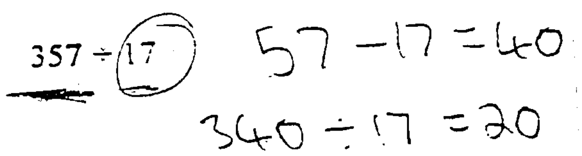
Can mental calculations involve paper and pencil?

Yes, paper and pencil can support mental calculation:

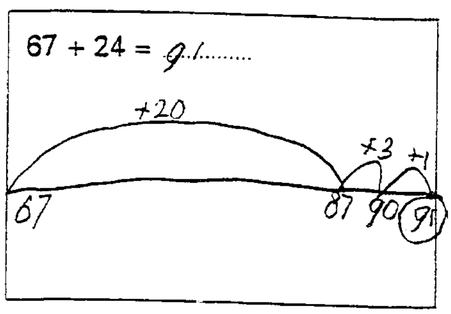
- through jottings – record of intermediate steps in a calculation;
- through images – models and diagrams that support the development of mental imagery;
- recording for another audience.

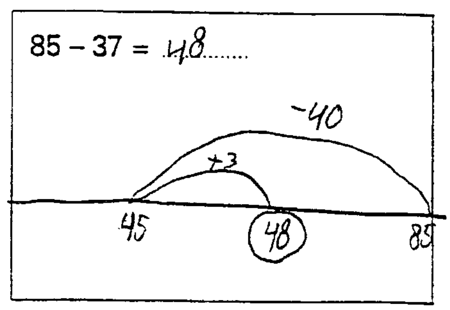
(i) Jottings

Examples of these are:



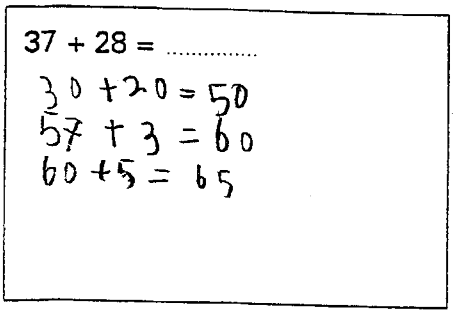
(ii) Images

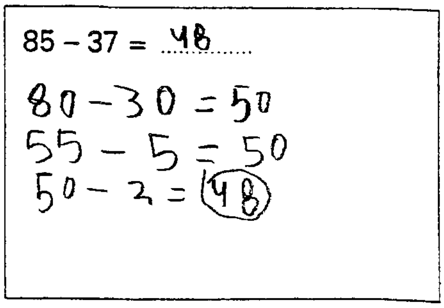




These images provide a visual representation of the way in which the calculation is being performed.

(iii) Recording for another audience





One further use of paper and pencil is in the extended recording of thought processes. This is a variation on getting children to explain their methods orally. A written account can help them begin to use appropriate notation and form the basis for developing more formal written methods.

Is speed important?

Once a strategy has been introduced to the children, there comes a time to encourage them to speed up their responses and either use more efficient strategies or expand their repertoire of known facts. Rather than pitch the children against each other, it is better to encourage them to compete with themselves, aiming to better their previous performance. A ‘prepared mental mathematics test’ can help children to monitor changes in performance over time.

The traditional model of a mental mathematics test is a set of unseen questions. An alternative is to give the children examples of the type of questions a little in advance, say 10 minutes. In this way they can think about the most efficient way to answer such questions. The purpose of this preparation time is not to try to commit answers to memory. The children should sort the questions into those they ‘know’ the answer to and those that they need to figure out. Pairs of children can talk about their figuring out methods and then the whole class can spend some time discussing the strategies used. Collecting the questions and then giving the children the test with the questions in a random order also encourages this attention to strategies. The same test can be used at a different time for children to try to beat their previous score.

One minute multiplication test

Warn the children in advance that they are going to be given a mental multiplication test. Classes or individuals can be set different challenges according to levels of attainment, so one class or child may be working on multiples of seven while another is working on multiples of three. The same 'blank' test paper can be duplicated for particular groups: ten randomly ordered 'multiplied by' questions:

x 3 =

x 7 =

x 4 =

and so on. The children have exactly one minute to write their particular number in each box and fill in the answer. Can they beat their previous best scores?

Other activities which encourage the development of quick response include:

Round the world

Invite one child to stand behind his or her neighbour's chair. Pose a mental mathematics question. The two children try to be first to answer. If the standing child correctly answers first, he or she moves on to stand behind the next person. If the sitting child answers first, the two swap places and the child now standing up moves on to stand behind the next person. Continue asking questions until a child is back to the starting point. Who moved most places? Can anyone get 'round the world', ie back to their original chair?

Beat the calculator

Invite pairs of children to the front and provide one of them with a calculator. Pose a calculation that you expect the children to be able to do mentally. The child with the calculator *must* use it to find the answer, even if they can mentally work more quickly. The winner is the best of five questions. The teacher usually has to be a strong referee to ensure the 'calculator' child does not call out the answer without keying in the calculation.

What might a session on mental calculation look like?

As most sessions on mental calculation involve class discussion, they need to be managed in a similar way to discussion sessions in, for example, science or geography lessons. The traditional method of asking a question and inviting a volley of hands to go up has several drawbacks:

- it emphasises the rapid, the known, over the derived – children who 'know' the answer beat those figuring it out;
- it is unhelpful to those figuring out – children collecting their thoughts are distracted by others straining to raise hands high and muttering 'Miss'.

Sessions on mental calculation therefore need to be managed in a way that enables all children to take part. Lessons need to be organised to provide some thinking time that enables rapid rather than instant responses and supports those children who need a bit longer to figure things out. Successful strategies include:

- insisting that *nobody* puts a hand up until the signal and silently counting to five or so before giving the signal;
- using digit cards for all children to show their answer at the same time;
- children simply raising a thumb to indicate that they are ready to answer.

Whatever activity is used, it is important to spend time discussing the various ways that children reached the answer, to point out the range of possible strategies and to highlight the most efficient and appropriate strategies. This can be reinforced by inviting individual children to the blackboard to show their jottings and give an oral explanation of their method or strategy. It is also important to encourage the children to commit more number facts to memory.

The National Numeracy Strategy recommends an initial five to 10 minutes at the beginning of each lesson to reinforce and practise some mental mathematics. This introductory session can be used by the teacher to get children to use a range of strategies recently and more distantly learned, and to apply the strategies to a variety of simple contextualised and non-contextualised problems. Such a session can be used to develop speed and flexibility of response and the learning of mathematical facts.

All of the mental calculation strategies discussed in this booklet will need to be taught by teachers and discussed with the children in a whole class group. Children will learn by comparing their strategies and discussing which strategies appear more effective for particular problems with particular types of number.

This focus on the teaching and learning of mental calculation strategies could form the basis of the main teaching activity, following the lesson plan described in the National Numeracy Strategy. However, at different stages of their schooling, children will be learning other aspects of mathematics, including recording their work and using correct mathematical vocabulary and notation.

There will be times when the teacher has to develop several aspects of the mathematics, either within the same lesson or from one lesson to the next. The *Framework for teaching mathematics* provides clear guidelines where parallel connected developments are taking place, moving from mental methods to informal, then to more formal written methods and back again, or when teaching children to use mathematical aids such as calculators for calculations where mental methods, or formal written methods alone, are inefficient or cumbersome.

Finally, the National Numeracy Strategy recommends that at the end of each lesson the teacher summarises the main theme of the lesson with the whole class. Again, this session provides an opportunity to consolidate key features of mental calculation strategies.

Part 3
Teaching addition and subtraction strategies

Contents

Each section contains activities for use with children.
Notes about the activities appear in blue.

Features of addition and subtraction

Numbers can be added in any order: it does not matter in which order two numbers are added. Take any pair of numbers, say 7 and 12, then 7 + 12 = 12 + 7.

When three numbers are added together, they can be taken in any order. In practice, two of the numbers have to be added together first, and then the third number is added to this intermediate number to give the result of the calculation. So, for example:

$$\begin{aligned} 7 + 5 + 3 &= (7 + 5) + 3 \\ &= 7 + (5 + 3) \\ &= (3 + 7) + 5 \end{aligned}$$

Order does matter in subtraction. Thus 5 – 3 is not the same as 3 – 5. But a series of subtractions can be taken in any order. For example,

$$15 - 3 - 5 = 15 - 5 - 3$$

Every addition calculation can be replaced by an equivalent subtraction calculation and vice versa. For example the addition

$$\begin{aligned} 5 + 7 &= 12 \\ \text{implies } 5 &= 12 - 7 \\ \text{and } 7 &= 12 - 5 \end{aligned}$$

In the same way

$$\begin{aligned} 13 - 6 &= 7 \\ \text{implies } 13 &= 7 + 6 \\ \text{and } 6 &= 13 - 7 \end{aligned}$$

Any numerical equivalence can be read from left to right or from right to left. So 6 + 3 = 9 is no different from 9 = 6 + 3

Counting forwards and backwards

Children first encounter the act of counting by beginning at one and counting on in ones. Their sense of number is extended by beginning at different numbers and counting forwards and backwards in steps, not only of ones, but also of twos, tens, hundreds and so on. The image of a number line helps them to appreciate the idea of counting forwards and backwards. They will also learn that, when adding two numbers together, it is generally easier to count on from the larger number rather than the smaller. Eventually 'counting-on' will be replaced by more efficient methods.

Expectations

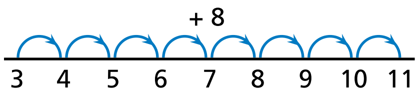
Year 1	4 + 8	**count on in ones from 4 or count on in ones from 8**
	7 – 3	**count back in ones from 7**
	13 + 4	**count on from 13**
	15 – 3	**count back in ones from 15**
	18 – 6	**count back in twos**
Year 2	14 + 3	**count on in ones from 14**
	27 – 4	**count on or back in ones from any two-digit number**
	18 – 4	**count back in twos from 18**
	30 + 3	**count on in ones from 30**
Year 3	40 + 30	**count on in tens from 40**
	90 – 40	**count back in tens from 90 or count on in tens from 40**
	35 – 15	**count on in steps of 3, 4, or 5 to at least 50**
Year 4	73 – 68	**count on 2 to 70 then 3 to 73**
	86 – 30	**count back in tens from 86 or count on in tens from 30**
	570 + 300	**count on in hundreds from 300**
	960 – 500	**count back in hundreds from 960 or count on in hundreds from 500**
Year 5	$1\frac{1}{2} + \frac{3}{4}$	**count on in quarters**
Year 6	1.7 + 0.5	**count on in tenths**

Activities

Get the children to count forward in ones, one after the other round the class. When you clap your hands, they then have to start counting backwards. On the next clap they count forwards, and so on.

You may wish to ensure that numbers do not go below zero, although the activity could easily be extended to include negative numbers. It can also be used for counting in twos, or in tens or hundreds.

Use empty number lines as an aid to counting forwards or backwards single-digit numbers, where there is no instant recall. Discuss the different methods children use and encourage them to move to more efficient methods. For example, they might see 3 + 8 as:



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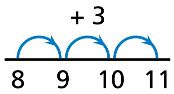

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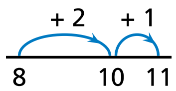
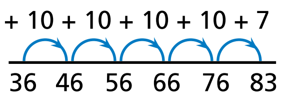
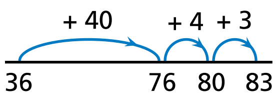


Empty (or blank) number lines provide a useful way for children to record their working and help you to see what method they are using. Discussion of the different lines can encourage children to move to a more efficient method.

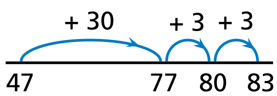
Get children to record a two-digit number on an empty number line. For example, 36 + 47 might be seen as counting on from 36 initially in steps of 10



or as by first counting on in a step of 40



or by reordering the calculation and then counting on from 47



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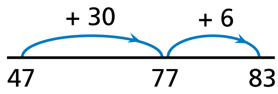


Discuss the different methods children use. Encourage them to move to a more efficient method, getting them to compare the number of steps involved and the time taken to calculate the answer.

Make a number line that goes up in tens, large enough for the whole class or a group to see. A chalk board, white board or overhead projector would be appropriate.

Ask an individual child to show where a given number, such as 26, would fit on the line. Ask other children to fit some numbers close to 26, such as 23 or 28. They may find that the original position of the number 26 needs to be adjusted. Invite them to adjust the position of each number until they are satisfied with them. Then get them to explain what they did to the rest of the group or class.

This activity encourages children to imagine where the numbers 1 to 9, 11 to 19 and 21 to 29 would appear on the line, and to count on mentally before they decide where to place the number they are given.

Tell the class that you are going to walk along an imaginary number line. You will tell them what number you are standing on and what size steps you are taking.

For example, ‘I am on 15 and am taking steps of 10.’ Invite them to visualise the number 15 on a number line and to tell you where you will be if you take one step forward (25). Take three more steps forward and ask: ‘Where am I now?’ (55). Take two steps back and ask: ‘Where am I now?’ (35), and so on.

Activities such as this help children to visualise counting on or back. The activity can be used for larger numbers. For example, tell them you are standing on 1570 and taking steps of 100 and ask them to visualise this. Then ask questions such as: ‘Where am I if I take two steps forward?...’

Reordering

Sometimes a calculation can be more easily worked out by changing the order of the numbers. The way in which children rearrange numbers in a particular calculation will depend on which number facts they have instantly available to them.

It is important for children to know when numbers can be reordered (eg 2 + 5 + 8 = 8 + 2 + 5 or 15 + 8 – 5 = 15 – 5 + 8 or 23 – 9 – 3 = 23 – 3 – 9) and when they can not (eg 8 – 5 ≠ 5 – 8).

The strategy of changing the order of numbers only really applies when the question is written down. It is difficult to reorder numbers if the question is presented orally.

Expectations

Year 1	2 + 7 = 7 + 2
	5 + 13 = 13 + 5
	3 + 4 + 7 = 3 + 7 + 4
Year 2	2 + 36 = 36 + 2
	5 + 7 + 5 = 5 + 5 + 7
Year 3	23 + 54 = 54 + 23
	12 – 7 – 2 = 12 – 2 – 7
	13 + 21 + 13 = 13 + 13 + 21 (using double 13)
Year 4	6 + 13 + 4 + 3 = 6 + 4 + 13 + 3
	17 + 9 – 7 = 17 – 7 + 9
	28 + 75 = 75 + 28 (thinking of 28 as 25 + 3)
Year 5	3 + 8 + 7 + 6 + 2 = 3 + 7 + 8 + 2 + 6
	25 + 36 + 75 = 25 + 75 + 36
	58 + 47 – 38 = 58 – 38 + 47
	200 + 567 = 567 + 200
	1.7 + 2.8 + 0.3 = 1.7 + 0.3 + 2.8
Year 6	34 + 27 + 46 = 34 + 46 + 27
	180 + 650 = 650 + 180 (thinking of 180 as 150 + 30)
	4.6 + 3.8 + 2.4 = 4.6 + 2.4 + 3.8
	8.7 + 5.6 – 6.7 = 8.7 – 6.7 + 5.6
	4.8 + 2.5 – 1.8 = 4.8 – 1.8 + 2.5

Activities

Present children with groups of four numbers that they are to add in their head. Ensure that, within each group of numbers, there are two numbers which are familiar totals to 10, for example:

8 + 3 + 5 + 2

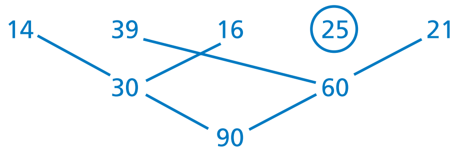
Discuss ways in which they did the addition and see if any of them chose to add 8 + 2 first and then add on the 5 + 3, or linked the 3 + 5 and added 8 + (3 + 5) + 2. Give them other similar examples and encourage them to look for pairs that add to make 10 or make doubles before beginning to add. Get them to make up similar examples for each other.

Children should learn that, when adding several numbers together, it is useful to look to see whether there are any pairs that can be matched to make an easy total, usually a multiple of 10.

Have regular short, brisk practice sessions where children are given ten questions such as 2 + 7 + 8 + 5 + 4 + 3 with at least five numbers such that some pairs total 10. Encourage children to time their responses, keep a personal record of their times and try to beat their personal best.

You could give children the same set of questions at regular intervals and encourage them to see how rapidly they can get to the answers. This should ensure that every child will see that they have made progress.

When children can find pairs of numbers that add to make multiples of 10, they can make use of this information when adding several numbers together. For example, when adding 14 + 39 + 16 + 25 + 21, it is sensible to pair numbers:



90 + 25 = 115

Children should learn that it is worth looking at all numbers that are to be added to see whether there are pairs that make convenient multiples of 10. The number tree shown in the diagram can be a helpful pictorial representation of the ways the numbers were paired.

In some particular sequences of numbers, the re-ordering strategy is useful and can provide opportunities for an investigative approach. For example:

Find quick ways of finding these answers:

1 + 2 + 3 + 4 + 5 + 6 = ?

5 + 7 + 11 + 13 = ?

3 + 6 + 9 + 12 + 15 + 18 = ?

1 + 2 + 3 + 4 + … + 98 + 99 = ?

Series of numbers such as these are always easier to add by matching numbers in pairs; in the first it is easier to add 1 + 6 = 7, 2 + 5 = 7, 3 + 4 = 7 and then to find 3 x 7. In the last example, by combining 1 + 99, 2 + 98 and so on up to 49 + 51, giving (49 x 100) + 50 or 4950.

Use a set of number cards, making sure that there are pairs that make multiples of 10. Divide the class into groups of three and give each child a card. Ask each group to add their numbers together. Encourage them to look for pairs of numbers to link together. List all the totals on a board or projector. Whose numbers give the largest total?

After each round, get each group to add all the three totals together. Check that the three totals add up to the same 'grand total'.

The numbers on the cards can depend on the knowledge of the children. For example they could be:

23	30	17	52	24	8	70	16
12	30	60	140	170	50	80	

You can arrange that each group gets cards that match their number skills.

The activity can be extended to using decimals, but in this case the aim is to make pairs that make a whole number: 1.4, 3.2, 0.6, 0.2, 1.6, 0.8, 2.3

Partitioning (i)
Using multiples of 10 and 100

It is important for children to know that numbers can be partitioned into, for example, hundreds, tens and ones, so that 326 = 300 + 20 + 6. In this way, numbers are seen as wholes, rather than as a collection of single-digits in columns. This way of partitioning numbers can be a useful strategy for addition and subtraction. Both numbers involved can be partitioned in this way, although it is often helpful to keep the first number as it is and to partition just the second number.

Expectations

Year 2	30 + 47	= 30 + 40 + 7
	78 – 40	= 70 – 40 + 8
	25 + 14	= 20 + 5 + 10 + 4
		= 20 + 10 + 5 + 4
Year 3	23 + 45	= 40 + 5 + 20 + 3
		= 40 + 20 + 5 + 3
	68 – 32	= 60 + 8 – 30 – 2
		= 60 – 30 + 8 – 2
Year 4	55 + 37	= 55 + 30 + 7
		= 85 + 7
	365 – 40	= 300 + 60 + 5 – 40
		= 300 + 60 – 40 + 5
Year 5	43 + 28 + 51	= 40 + 3 + 20 + 8 + 50 + 1
		= 40 + 20 + 50 + 3 + 8 + 1
	5.6 + 3.7	= 5.6 + 3 + 0.7
		= 8.6 + 0.7
	4.7 – 3.5	= 4.7 – 3 – 0.5
Year 6	540 + 280	= 540 + 200 + 80
	276 – 153	= 276 – 100 – 50 – 3

Activities

Use a die marked: 1, 1, 10, 10, 100, 100 for the game ‘Target 500’ with a group of children.

Each player may roll it as many times as they wish, adding the score from each roll and aiming at the target of 500. They must not ‘overshoot’. If they do, they go bust!

For example, a sequence of rolls may be:

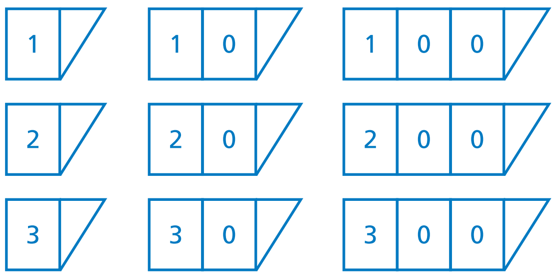
10, 10, 1, 100, 1, 100, 100, 1, 1, 10, 100

At this point, with a total of 434, a player might decide not to risk another roll (in case 100 is rolled) and stop, or to hope for another 10.

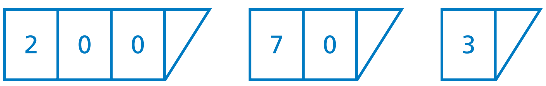
Who gets nearest to 500?

This game provides useful practice of building up numbers by mental addition using ones, tens and hundreds.

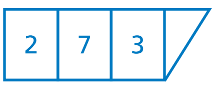
Use place value cards 1 to 9, 10 to 90 and 100 to 900:



Ask the children to use the cards to make a two-digit or a three-digit number by selecting the cards and placing them on top of each other. For example, to make 273, the cards



can be placed over each other to make



This could be used as a class activity, in which case you could ask individual children to select the appropriate card and to put it in the correct place. Alternatively it could be used with individuals as a diagnostic task, to check whether they understand place value in this context.

With a group of three to five players, play a game using the place value cards 1 to 9, 10 to 90 and 100 to 900.

Deal these 27 place value cards between the players.

Give the children a set of cards containing two- and three-digit numbers to be used as target numbers; for example:

153	307	682	914	530	890
745	201	26	79	468	96

Place these numbers in a pile and turn them over, one by one, to set a target number.

Player A inspects his or her cards to see if he or she has any part of that number and if so places it on the table.

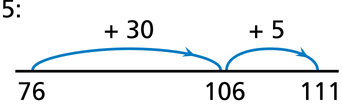
Play continues anticlockwise round the group and player B checks to see whether he or she has another part of the number, followed by player C, D, and so on. Whoever completes the target number keeps it.

Who wins the most target numbers?

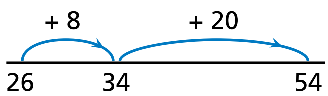
Games such as these can provide motivation for practice in partitioning numbers into their component hundreds, tens and ones. They can also provide the opportunity for children to learn from each other.

Use the empty number line to add or subtract two-digit numbers:

eg 76 + 35:



or 54 – 28:

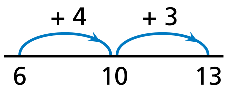


The empty number lines are a useful way of recording how children use multiples of 10 or 100 to add or subtract, and so provide a means for discussing the different methods that they use and to encourage them to move to more efficient methods.

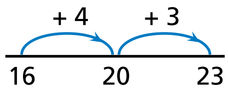
Partitioning (ii)
Bridging through multiples of 10

An important aspect of having an appreciation of number is to know when a number is close to 10 or a multiple of 10: to recognise, for example, that 47 is 3 away from 50, or that 96 is 4 away from 100. When adding or subtracting mentally, it is often useful to make use of the fact that one of the numbers is close to 10 or a multiple of 10 by partitioning another number to provide the difference. The use of an empty number line where the multiples of 10 are seen as 'landmarks' is helpful and enables children to have an image of jumping forwards or backwards to these 'landmarks'. For example,

6 + 7 = 6 + 4 + 3



In the case of subtraction, bridging through the next 10 or multiple of 10 is a very useful method (often termed 'shopkeeper's subtraction'; it is the method used almost universally with money). So the change from £1 for a purchase of 37p is carried out thus: '37 and 3 is 40 and 10 is 50 and 50 is £1'. The use of actual coins, or the image of coins, helps to keep track of the subtraction. The empty number line can provide an image for this method when the subtraction does not involve money. The calculation 23 – 16 can be built up as an addition:



'16 and 4 is 20 and 3 is 23, so add 4 + 3 for the answer.'
A similar method can be applied to the addition and subtraction of decimals, but here, instead of building up to a multiple of 10, numbers are built up to a whole number or to a tenth.
So 2.8 + 1.6 can be turned into 2.8 + 0.2 + 1.4 = 3 + 1.4

Expectations

Year 2	6 + 7 = 6 + 4 + 3
	23 – 9 = 23 – 3 – 6
	15 + 7 = 15 + 5 + 2
Year 3	49 + 32 = 49 + 1 + 31
Year 4	57 + 14 = 57 + 3 + 11 or 57 + 13 + 1
Year 5	3.8 + 2.6 = 3.8 + 0.2 + 2.4
	5.6 + 3.5 = 5.6 + 0.4 + 3.1
Year 6	296 + 134 = 296 + 4 + 130
	584 – 176 = 584 – 184 + 8
	0.8 + 0.35 = 0.8 + 0.2 + 0.15

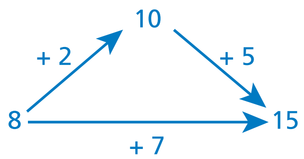
Activities

Show the class a single-digit number and ask a child to find its complement to 10. Repeat this many times, encouraging the children to respond as quickly as they can. Then offer two-digit numbers and ask for complements to 100.

Activities such as these provide practice so that children can acquire rapid recall of complements to 10 or 100.

Give the class two single-digit numbers to add. Starting with the first (the larger), ask what needs to be added to make 10, then how much more remains to be added on. Show this on a diagram like this:

Example: 8 + 7 = 8 + 2 + 5 = 10 + 5 = 15



Such diagrams provide a useful means of recording how the starting number is built up to 10 and what remains to be added. Children can be given blank diagrams and asked to use them to add sets of numbers less than 10.

Use examples with money, asking children to show what coins they would use to build up to the next convenient amount:

A packet of crisps costs 27p. How much change do you get from 50p?

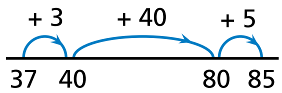
27p and 3p is 30p. Another 20p makes it up to 50p. The change is 3p + 20p = 23p.

This is the natural way to find a difference when using money, the coins providing a record of how the change was given. This method can usefully be applied to other instances of subtraction.

Use empty number lines for addition and subtraction, using multiples of 10 as interim numbers:

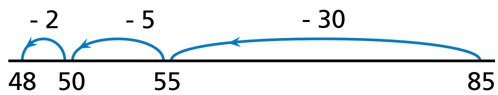
Example: 85 – 37.

(i) by adding on from 37 to 85 (the shopkeepers' method):



'37 and 3 makes 40 and 40 make 80 and 5 makes 85. So add 3 + 40 + 5 to get the answer'

(ii) by counting backwards:



In the first example, the next multiple of 10 is a first 'landmark' from the starting number, so 37 is built up to 40. Subsequent landmarks might be an intervening multiple of 10 (80 in the first case). In the second example, the first landmark from 85, the starting number, is 55, then the next is 50. Encouraging children to use number lines in this way provides a mental image that can assist with mental calculations.

Write a set of decimal numbers on the board, such as:

3.6, 1.7, 2.4, 6.5, 2.3, 1.1, 1.5, 1.8, 2.2, 3.9

and ask children, in turns, to find pairs that make a whole number.

Then extend the activity to finding pairs of numbers that make a whole number of tenths:

0.07, 0.06, 0.03, 0.05, 0.04, 0.05, 0.09, 0.01

The first activity with decimals is to build up to whole numbers, so 3.6 is added to 2.4 to make 6. In the case of hundredths, numbers such as 0.06 can be added to 0.04 to make 0.1.

Partitioning (iii) Compensating

This strategy is useful for adding numbers that are close to a multiple of 10, for adding numbers that end in 1 or 2, or 8 or 9. The number to be added is rounded to a multiple of 10 plus a small number or a multiple of 10 minus a small number. For example, adding 9 is carried out by adding 10 and then subtracting 1, and subtracting 18 is carried out by subtracting 20 and adding 2. A similar strategy works for decimals, where numbers are close to whole numbers or a whole number of tenths. For example, 1.4 + 2.9 = 1.4 + 3 – 0.1 or 2.45 – 1.9 = 2.45 – 2 + 0.1

Expectations

Year 1	5 + 9 = 5 + 10 – 1
Year 2	34 + 9 = 34 + 10 – 1
	52 + 21 = 52 + 20 + 1
	70 – 9 = 70 – 10 + 1
Year 3	53 + 11 = 53 + 10 + 1
	58 + 71 = 58 + 70 + 1
	84 – 19 = 84 – 20 + 1
Year 4	38 + 69 = 38 + 70 – 1
	53 + 29 = 53 + 30 – 1
	64 – 19 = 64 – 20 + 1
Year 5	138 + 69 = 138 + 70 – 1
	405 – 399 = 405 – 400 + 1
	$2\frac{1}{2} + 1\frac{3}{4} = 2\frac{1}{2} + 2 - \frac{1}{4}$
Year 6	5.7 + 3.9 = 5.7 + 4.0 – 0.1

Note to teachers

The Framework for teaching mathematics refers to this strategy as 'Rounding and compensating', whereas in this guidance 'Compensating' alone has been used. This is because adding 11 as (10 + 1) or 32 as (30 + 2) are further examples of 'Partitioning', which is a major organisational strategic principle. Adding 9 as (10 – 1) or 18 as (20 – 2) is clearly a variation on this same theme, so such examples are included as partitions in a special sense.

In the same spirit, 'Using near doubles' in the next section is also included in the general theme of 'Partitioning'.

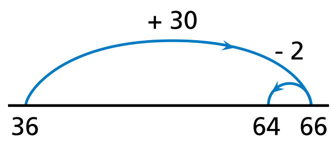
Activities

Use a class shop for children to buy items that cost about 20p. Make sure that some items are marked 9p, 11p, 19p and 21p. Suggest shopping lists and ask individual children, in turn, to find their totals. For example, buy items costing 7p and 9p. Think of 9p as 1p less than 10p, so the total of 7p and 9p is 1p less than (7p + 10p), ie 16p.

Children might be familiar with the fact that many goods in the shops have prices such as 99p or £3.99 and you might like to discuss why this is so. When adding any amount such as 9p, 19p, 29p, etc, encourage children to think of them as 10p – 1p, 20p – 1p and so on, and to employ the fact that it is easy to add a multiple of 10.

Activities

Use a number line to carry out additions such as 36 + 28 by counting forward 30 and compensating by counting backwards by 2:



Ask them to visualise a number line to show 45 + 29, 27 + 39…

The number line is a means of showing how the process of counting forward and then back works. It can also be a useful way of getting children to visualise similar examples when working mentally.

Prepare two sets of cards for a subtraction game. Set A has numbers from 12 to 27. Set B contains only 9 and 11 so that the game involves subtracting 9 and 11. Shuffle the cards and place them face down.

The children also each need a playing board:

15	3	9	16
5	18	4	17
13	7	12	8
6	11	14	10
18	1	2	17

The children take turns to choose a number from set A and then one from set B. They subtract the number from set B from the one from set A and mark the answer on their board. The first person to get three numbers in a row on their board wins.

Discuss how they carried out the subtraction. Encourage them to use the strategy of compensating.

Games such as these can provide motivation to practise the strategy of rounding up to a multiple of ten and then compensating by subtracting. By changing the numbers in set A, negative numbers can also be used.

Use a number square for adding tens and numbers close to 10. To find 36 + 28, first find 36 + 30 by going down three rows, then compensate by going back along that row two places:

31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70

So 36 + 28 = 64.

First, children need to know that they can use the table to add 10 to any number by moving down to the number below it. (36 + 10 = 46, which is just below 36, and 36 + 20 is 56, to be found in the second row below.) Similarly, subtracting ten is performed by moving to numbers in the row above (57 – 10 = 47). Then they can use this strategy for adding or subtracting numbers that are close to a multiple of 10 by finding the correct row and then moving to the right or the left.

Prepare a set of cards with numbers such as 3.9, 2.9, 5.1, 4.1, 5.8, 3.2 … and work with a group of children.

3.9	2.9	5.1	4.1	5.8	3.2

They should take turns to choose a single-digit number, to turn over one of the prepared cards and then add these two numbers together. Get them to tell the others in the group how they carried out the addition. Encourage them to see, for example, 7 + 4.1 as 7 + 4 + 0.1 and 7 + 3.9 as 7 + 4 – 0.1

This could be extended so that children choose a number with one decimal place as the starting number. So, for example, 2.4 + 3.9 = 2.4 + 4 – 0.1.

Provide children with practice of examples such as 264 – 50, 3450 – 300. Then ask them to subtract numbers such as 9, 99, 999 or 11, 31, 91 and so on.

Encourage them to use a method of rounding and compensating, so 264 – 39 = 264 – 40 + 1 and 2500 – 99 = 2500 – 100 + 1

This activity could be used as a class activity, asking individual children round the class to supply the answer. Encourage them to explain how they carried out the subtraction.

Partitioning (iv)
Using near doubles

If children have instant recall of doubles, they can use this information when adding two numbers that are very close to each other. So, knowing that 6 + 6 = 12, they can be encouraged to use this to help them find 7 + 6, rather than use a ‘counting on’ strategy or ‘building up to 10’.

Expectations

Year 1	5 + 6	**is double 5 and add 1 or double 6 and subtract 1**
Year 2	13 + 14	**is double 14 and subtract 1 or double 13 and add 1**
	40 + 39	**is double 40 and subtract 1**
Year 3	18 + 16	**is double 18 and subtract 2 or double 16 and add 2**
	36 + 35	**is double 36 and subtract 1 or double 35 and add 1**
	60 + 70	**is double 60 and add 10 or double 70 and subtract 10**
Year 4	38 + 35	**is double 35 and add 3**
	160 + 170	**is double 150 and add 10 then add 20, or double 160 and add 10, or double 170 and subtract 10**
	380 + 380	**is double 400 and subtract 20 twice**
Year 5	1.5 + 1.6	**is double 1.5 and add 0.1 or double 1.6 and subtract 0.1**
Year 6	421 + 387	**is double 400 add 21 and then subtract 13**

Activities

Choose a double fact and display it on the board – for example:

8 + 8

Invite someone to give the total. Then ask for suggestions of addition facts that the children can make by changing one of the numbers, for example:

8 + 9, 7 + 8

Then repeat the activity by giving the children double facts that they might not know:

17 + 17 = 34   28 + 28 = 56   136 + 136 = 272

Ask them to say how they could work out some near doubles, such as:

17 + 18 or 16 + 17 or 27 + 28 or 136 + 137

Invite the children, in turn, to provide their own double fact and ask other children to suggest some addition facts that they can generate from it.

Children from year 2 should know the doubles up to at least 10 + 10. Giving them, later, a double that they could not reasonably be expected to know and asking them to calculate some related facts enables you to check that they are using the strategy.

Working with a whole class, ask one child to choose a number less than 10. Then ask them, in turns, to double the number, then double the result and so on. Get them to write the numbers on the board or on an overhead projector. For example:

3 6 12 24 48 96…

Challenge them to see how far they can go with this doubling sequence.

Then ask them to produce another sequence by starting with a number, then doubling it and adding 1 each time. So starting again with 3, say, they would get:

3 7 15 31 63…

Other variations would be to use the rules 'double minus 1' or 'double plus 2'.

Being proficient at the process of doubling is essential if children are to use the strategy of finding 'near doubles'. Asking children to produce the next term in a sequence ensures that they must all take note of all the answers that are given by other children.

Play 'Think of a number' using a rule that involves doubling and adding or subtracting a small number. For example,

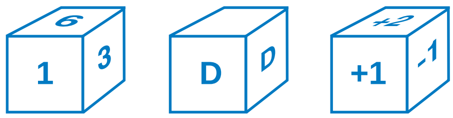
'I'm thinking of a number. I doubled it and added 3.

My answer is 43. What was my number?'

Then invite the children to invent some similar examples themselves.

All such 'Think of a number' activities require children to 'undo' a process, in this case the process of doubling and adding.

This is a game for a group of children, and needs three dice. One is numbered 1 to 6; a second has four faces marked with a D for 'double' and two blank faces; the third is marked +1, +1, +1, +2, -1, -1.



Children take turns to throw the three dice and record the outcome.

They then decide what number to make.

For example, if they throw 3, D and +1 they could: double the 3 then add 1 to make 7, or they could add 1 to the 3 to make 4 and then double 4 to make 8.

What is the smallest possible total?
What is the biggest?
What totals are possible with these three dice?
Which totals can be made the most ways?

Games such as this one can provide motivation for children to practise the strategy and questions such as: 'What totals are possible?' Encourage children to reflect on the processes rather than just to find one answer.

Get children to practise adding consecutive numbers such as 45 and 46. Then give the children statements such as: 'I add two consecutive numbers and the total is 11.' Ask them: 'What numbers did I add?'

Knowing doubles of numbers is useful for finding the sum of consecutive numbers. The reverse process is more demanding.

Partitioning (v)
Bridging through numbers other than 10

Time is a universal measure that is non-metric, so children need to learn that bridging through 10 or 100 is not always appropriate. A digital clock displaying 9.59 will, in two minutes time, read 10.01 not 9.61. When working with minutes and hours, it is necessary to bridge through 60 and with hours and days through 24. So to find the time 20 minutes after 8.50, for example, children might say 8.50 + 10 minutes takes us to 9.00, then add another 10 minutes.

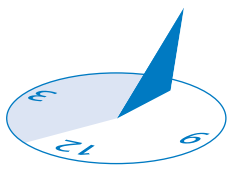
Expectations

Year 1	1 week = 7 days
	What time will it be in one hour's time?
	How long is it from 2 o'clock to 6 o'clock?
	It is half past seven. What time was it 3 hours ago?
	It is 7 o'clock in the morning. How many hours to mid-day?
Year 2	1 year = 12 months
	1 week = 7 days
	1 day = 24 hours
	1 hour = 60 minutes
	What time will it be 1 hour after 9 o'clock?
	10.30 to 10.45
	9.45 to 10.15
Year 3	40 minutes after 3.30
	50 minutes before 1.00 pm
	It is 10.40. How many minutes to 11.00?
	It is 9.45. How many minutes to 10.00?
Year 4	It is 8.35. How many minutes to 9.15?
Year 5	It is 11.30. How many minutes to 15.40?
Year 6	It is 10.45. How many minutes to 13.20?

Activities

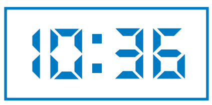
Choose a sunny day. Set up a shadow clock in the playground. Set a timer to indicate the hours. Draw the attention of the class to the fact that is now 9 o'clock or 10 o'clock, and so on.

At each hour, get someone to go out and mark in the position of the shadow and to record the length of the shadow. Record the name of the person who does the measuring and the appropriate time. Repeat this every hour. At the end of the day, discuss the times, emphasising that the next time after 12.00 is 1.00pm (or 13.00 if you want to discuss the 24-hour clock.)



This activity helps to focus attention on hourly intervals and that 12.00 or midday is an important time because there are 24 hours in a day.

Have a digital clock in the classroom.



Get the class to look at it at various times of the day and ask: 'How many minutes it is to the next hour (or next 'o-clock')?'

Encourage the children to count on from 36 to 40, then 50 and then 60, to give 24 minutes.

Then ask questions such as: 'How long will it be to 11.15?' Get them to count on to 11.00 and then add on the extra 15 minutes?.

Digital clocks could suggest to children that minutes behave like ordinary numbers, so that they might count on 59, 60, 61 and so on, not realising that at 60 the numbers revert to zero as a complete hour is reached. It therefore helps if you draw children's attention to what happens to the clock soon after, say, 9.58 and to stress the difference between this and what happens in other digital meters such as electricity meters or the meters that give the distance travelled by a bicycle or car.

Give the class, or a group of children, statements such as:

'Jane leaves home at 8.35am and arrives at school at 9.10am. How long is her journey?'

Discuss how they can find the answer and discuss their various methods, writing each on the board.

Some might say:

'8.35, 8.40, 8.50, 9.00, 9.10,'

counting 5 and 10 and 10 and 10 to give the total time.

Others might say:

'8.35 and 25 minutes takes us to 9.00, so add on another 10 minutes.'

Children need to remember that, for minutes, they need to count up to 60 before getting to the next hour. Some children might be tempted to say 8.35, 8.40, 8.50, 8.60 and so on, expecting to go on until they get to 100. Reference to a clock face should help them to see why this is not appropriate.

Use local bus or train timetables.

This is part of a train timetable from Bristol to London:

Bristol	London
08.30	09.45
10.15	11.40
12.30	13.55
15.15	16.48

Ask questions such as 'How long does the 8.30 train take to get to London?' Encourage children to count up to 9.00 and then to add on the extra 45 minutes.

Ask: 'Which train takes the shortest time?' 'Which takes the longest?'

Timetables are often difficult to read, so extracts such as this one are helpful. Suggest that children build the starting times up to the next hour, and then add on the remaining minutes.

Activities

Get the children to plan a journey such as this one:

Coaches arrive and leave Alton Towers at these times:

	Arrive	Leave
Coach A	08.00	14.30
Coach B	09.30	15.45
Coach C	10.15	16.00
Coach D	11.45	17.30

Ask questions such as: 'Which coach gives you the most time at Alton Towers?' 'Which gives you the least?'

Discuss the strategies that children use to find the times. For Coach C, for example, some might bridge 10.15 up to 11.00 and then find the number of remaining hours; others might bridge from 10.15 through 12.15 to 15.15, counting in hours and then add on the remaining 15 minutes. They need to remember that they need to count 10, 11, 12 and then 1, 2 and so on.

Part 4
Teaching multiplication and division skills and strategies

Contents

Each section contains activities for use with children.
Notes about the activities appear in blue.

Features of multiplication and division

Three different ways of thinking about multiplication are:

- as repeated addition, for example 3 + 3 + 3 + 3;
- as an array, for example four rows of three objects;
- as a scaling factor, for example making a line three times longer.

The use of the multiplication sign can cause difficulties. Strictly, 3 x 4 means 3 multiplied by 4 or four 'lots of' three. This is counter to the intuitive way of interpreting 3 x 4, which is often thought of as three lots of four. Fortunately, multiplication is commutative, 3 x 4 being equal to 4 x 3, so the outcome is the same. The colloquial use of 'three times four' provides another confusion: a phrase that was derived, presumably, from the idea of 'three, taken four times' – or four taken three times.

When multiplication and addition or subtraction are combined, as in 3 x (4 + 5), the fact that 3 x (4 + 5) = (3 x 4) + (3 x 5), can be useful in mental calculation.

Division and multiplication are inverse operations. However, whereas any two whole numbers can be multiplied to make another whole number, this is not always the case for division. 12 ÷ 4 gives a whole number, but 12 ÷ 5 does not. For mental calculation it is important to know multiplication facts in order that the related division facts can be worked out.

Because knowledge of multiplication and the corresponding division facts up to 10 x 10 is so important, the first two sections in this chapter concentrate on how children acquire these facts.

Knowing multiplication and division facts to 10

Instant recall of multiplication and division facts is a key objective in developing children's numeracy skills. The expectation (see below) is that most year 5 children know the multiplication facts up to 10 x 10 and that by year 6 most children will also know the associated division facts.

However, learning these facts and being fluent at recalling them quickly is a gradual process which takes place over time and which relies on regular opportunities for practice in a variety of situations.

The ability to work out and knowing by heart are linked and support each other. For example, the child who can work out the answer to 6 x 8 by recalling 6 x 2 and then doubling this result twice will, through regular use of this strategy, become more familiar with the fact that 6 x 8 is 48. In the interest of speed and accuracy, it is important that these facts are known by heart, even if children are able to employ effective strategies for working them out.

Expectations

Year 1	Count in twos – 2, 4, 6, 8, … to 20
	Count in tens – 10, 20, 30 … to 50
	Count in fives – 5, 10, 15, 20, … to 20 or more
Year 2	Count in fives – 5, 10, 15, 20, … to at least 30
	Recall the 2 times table up to 2 x 10
	Recall the 10 times table up to 10 x 10
	Recall division facts for the 2 and 10 times tables
Year 3	Count in threes – 3, 6, 9, 12, … to 30
	Count in fours – 4, 8, 12, 16, … to 40
	Recall the 5 times table up to 5 x 10
	Recall the corresponding division facts
Year 4	Count in sixes, sevens, eights and nines
	Recall the 3 times table up to 3 x 10
	Recall the 4 times table up to 4 x 10
	Recall the corresponding division facts
Year 5	Know the square numbers (eg 2 x 2, 3 x 3, 4 x 4, etc) up to 10 x 10
	Recall the 6 times table up to 6 x 10
	Recall the 8 times table up to 8 x 10
	Recall the 9 times table up to 9 x 10
	Recall the 7 times table up to 7 x 10
	Recall the corresponding division facts
Year 6	Know the squares of 11 and 12 (ie 11 x 11 and 12 x 12)

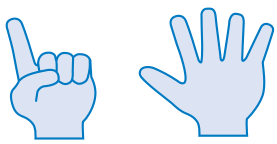
Activities that support and develop children's ability to work out facts that they cannot recall

As children begin to be able to recall certain multiplication facts, they should be encouraged to develop strategies that allow them to work out other facts from the ones they know.

Children who are able to count in twos, fives and tens can use this knowledge to work out other facts such as 2 x 6, 5 x 4, 10 x 9.

Show children how to hold out the appropriate number of fingers and, touching each one in turn, to count in twos, fives or tens.

For 2 x 6, hold up 6 fingers:

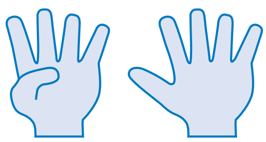


As children touch each of the six fingers in turn, they say '2, 4, 6, 8, 10, 12' to get the answer 12.

For 5 x 4 , hold up four fingers:

This time they say '5, 10, 15, 20' to get the answer 20.

For 10 x 9, hold up nine fingers:

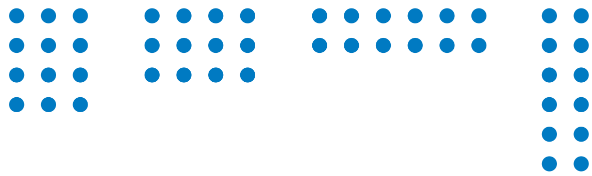


Count '10, 20, 30, 40, 50, 60, 70, 80, 90' to give the answer 90.

The process of touching the fingers in turn acts as a means of keeping track of how far the children have gone in creating a sequence of numbers. The physical action can later be visualised without any actual movement.

Discuss ways of grouping a number of dots in a rectangular array:

For example, 12 can be represented as follows:



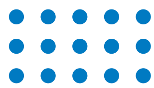
Children can use this idea to play this game:

Player A takes a handful of counters, counts them and announces to player B how many there are.

Player B then says how he or she can make this into a rectangular array and then proceeds to make the array with the counters. If they both agree it is correct, player B gets a point. (Single line arrays are not allowed in this game.)

Both players record the multiplication fact that this represents.

For example, player A takes 15 counters. Player B says, 'I can make three lots of five,' and proceeds to arrange the counters:



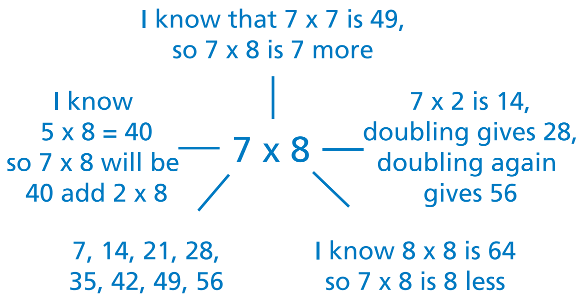
They record their result as 15 = 3 x 5 or 15 = 5 x 3 or 3 x 5 = 15 or 5 x 3 = 15.

(At the end of the game, discuss the numbers that could not be made into a rectangular array, ie the prime numbers.)

The process of arranging counters in a rectangular array is a helpful introduction to understanding about factors. If a number can be arranged in a rectangle (excluding a straight line) then it can be factorised. Numbers that can only be so arranged as a straight line are prime numbers.

Activities that support and develop children's ability to work out facts that they cannot recall

Write a multiplication in the middle of the board. Ask the children to come up to the board and say how they would figure out the result. Record their different methods and use them as a basis for discussion.

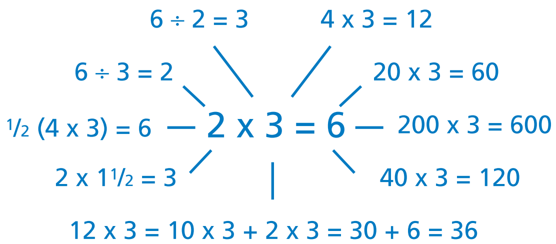


Getting children to tell others how they carried out their multiplication is a way of allowing them to consolidate their own understanding. It also allows other children to be introduced to methods that they might not have thought of for themselves.

Write a multiplication fact in the middle of the board and ask the children:

'Now that we know this fact, what other facts do we also know as a result of this one?'

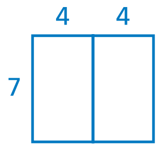
Encourage children to come to the board, explain their idea and then write it on the board.



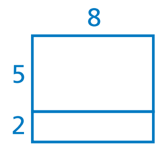
As in the last activity, there is opportunity for the children who are explaining to others to clarify their own thoughts and also for the others to be introduced to different methods.

Use the empty rectangle as a device for showing multiplication and division facts that can be worked out from known facts. For example:

7 x 8 is 7 x 4 and another 7 x 4:



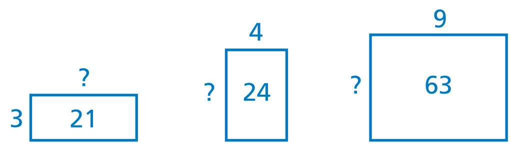
7 x 8 is 5 x 8 and 2 x 8:



Use the same method for bigger numbers, for example, 23 x 4 is 20 x 4 and 3 x 4.

These diagrams are, in effect, an area model for multiplication, a model that is also useful later when studying multiplication in algebra. They provide a visual means of showing the way that multiplication can be split up into two or more parts.

Use rectangles to practise division:



21 ÷ 3 = ? 24 ÷ 4 = ? 63 ÷ 9 = ?

The same model that was used in the previous example can be used for division, and helps to show that multiplication and division are inverse operations.

Show the children all the 10 x 10 multiplication facts displayed in a table:

x	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Encourage them to highlight the facts which they already know (eg multiplication by 1, 2, 5 and 10).

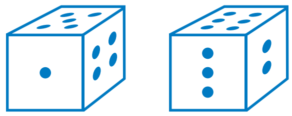
Then ask them to identify the facts that they can work out easily, eg the 4 times table is twice the 2 times table, the 8 times table is twice the 4 times table, the 9 times table is one multiple less than the 10 times table (ie 6 x 9 is 6 less than 6 x 10, and so on).

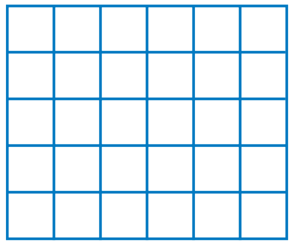
Then get them to identify the 'tricky' ones that they need to work on, for example 7 x 6.

Multiplication squares display the patterns that exist in the tables. Children might like to keep their own personal square and to colour in all the facts that they are sure of, leaving it obvious to them which ones they still need to learn. Then, as these are learnt, they can be coloured in.

Play a game for two or three children:

Each child throws a die twice to obtain the length and width of a rectangle. They then draw the rectangle on squared paper, find its area and see whose rectangle is the largest one.





This activity uses an area model for multiplication. You could ask questions such as:
How did you work that out?
Did you split the rectangle up in any way to help you?
What multiplication have you worked out there?
Can you give me a division fact that comes from that?
What if the number on that side of the rectangle was twice as big (or 10 more, or 10 times bigger, or …)?

Activities that develop quick recall

It is important for children to have quick recall of the multiplication facts. Children need a great deal of practice if they are to have recall of the facts up to 10 x 10. Generally, frequent short sessions that focus on the multiplication facts are more effective than longer sessions occurring less frequently. The practice should involve as wide a variety of activities as possible. Some suitable ones are suggested below.

Distribute a number of cards with a multiplication fact with one number missing, such as:

? x 4 = 24	5 x ? = 35

The children need to place their cards on the space that gives the missing number on a sorting tray like this:

2	3	4
5	6	7
8	9	10

The children should time themselves to see how long it takes them to sort all nine cards correctly. Then they can repeat the activity several times over the next week or so, keep a record of the time it took and see if they can improve on their time.

Place tracing paper over a random collection of numbers such as:

3	4	9
8	7	8
1	2	3
2	6	8
5	4	9
9	6	7

To practise recalling multiples of a given number, eg multiples of four, children write the appropriate multiple over each printed number.

12	16	36
32	28	32
4	8	12
8	24	32
20	16	36
36	24	28

How many can they do in two minutes?

This time the children are given a fixed time to see how many multiplication facts they know. The activity can then be repeated several times over the next few days, keeping a record of how far they get. They can then see how their speed of recall improves.

Design a set of cards containing questions and answers where the answers are not answers to the question above it but they are answers to another question on a different card.

3 x 5	6 x 7	4 x 4	9 x 8	8 x 5
40	15	42	16	72

Distribute the cards around the class so that everyone has at least one (usually two or three) and ask one child to read out the multiplication at the top of their card. The child who has the correct answer reads it out and then reads out the question at the top of that card. This continues until all cards have been used up.

The game works best if the cards form a loop where the question on the last card is linked to the answer on the first, as in the five cards above.

One of the benefits of activities like this is that every child has to work out all the answers to see if they have the one that is asked for. So every child gets plenty of practice.

Have a series of cards (related to one of the multiplication tables) which show a multiplication on one side and the answer on the other. Children play a game on their own where they lay the cards out in front of them with either all the multiplications or all the results showing.

1 x 7	7 x 9	4 x 7	9 x 7
5 x 7	6 x 7	2 x 7	6 x 7
7 x 6	3 x 7	10 x 7	7 x 5

The player touches a card, says what is on the other side and then turns it over. This continues until all the cards are turned over. If the answer to any card is wrong, it has to be turned back over and another card tried.

Children should keep a record of the time taken to complete the cards for any times table and they should try to improve on this next time.

Show children this grid and explain that it contains the numbers that appear in the 2, 5 and 10 times tables. Some numbers appear in more than one table.

4	6	8
10	12	14
15	16	18
20	25	30
35	40	45
50	60	70
80	90	100

Point to a number and invite the children to say what the multiplication fact is. Encourage a quick response.

Do the same activity with the x 3, x 4, x 6, x 7, x 8 and x 9 multiplication tables, using this chart:

9	12	16	18
21	24	27	28
32	36	42	48
49	54	56	63
64	72	81	

This activity requires children to recognise the product of two numbers and to say what those numbers are. They need to ask themselves: 'How did I get the answer 42 (say)?' 'What two numbers do I multiply to get that answer?' This shows up the relation between multiplication and division.

Multiplying and dividing by multiples of 10

Being able to multiply by 10 and multiples of 10 depends on an understanding of place value and is fundamental to being able to multiply and divide larger numbers.

Expectations

Year 2	7 x 10
	60 ÷ 10
Year 3	6 x 100
	26 x 10
	700 ÷ 100
Year 4	4 x 60
	3 x 80
	351 x 10
	79 x 100
	976 x 10
	580 ÷ 10
Year 5	9357 x 100
	9900 ÷ 10
	737 ÷ 10
	2060 ÷ 100
Year 6	23 x 50
	637.6 x 10
	135.4 ÷ 100

Activities

Use function machines that multiply by 10:

Enter a number → x 2 → x 10 = ?

Enter the same number → x 10 → x 2 = ?

What do you notice?

Try some divisions:

40 → ÷ 4 → ÷ 10 = ?

40 → ÷ 10 → ÷ 4 = ?

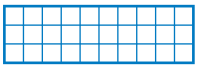
Try other starting numbers, such as 60, 20, 80 and so on.

The function 'machine' is a useful device for focusing attention on particular operations, in this case multiplication and then division. In the first part, children will notice that the order of multiplication does not matter – the effect of multiplying by 10 and then by 2 is the same as multiplying by 2 and then by 10. The machines can also work backwards, again illustrating that multiplication and division are inverse operations. But when dividing, unless the children are able to cope with decimal fractions, you will need to be careful which numbers are entered.

Activities

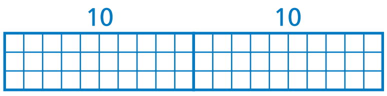
Use a rectangular array to show multiplication by 10.

For example, 3 x 10:

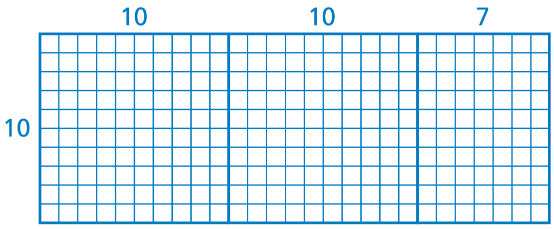


How many in each row? How many rows? How many altogether?

Similarly, for 3 x 20:



and for 27 x 10:



The area model is a useful one, especially to show how a number can be partitioned into tens. Children can, later, visualise the image as an aid to mental calculation.

Use the multiplication chart:

1	2	3	4	5	6	7	8	9
10	20	30	40	50	60	70	80	90
100	200	300	400	500	600	700	800	900
1000	2000	3000	4000	5000	6000	7000	8000	9000

Discuss the fact that the numbers on each row are found by multiplying the number above them by 10. So:

8 x 10 is 80, 40 x 10 is 400, and 500 x 10 is 5000.

If you skip a row, the numbers are multiplied by 100, so:

2 x 100 is 200, 70 x 100 is 7000.

Use the chart for dividing:

50 ÷ 10 = 5, 600 ÷ 10 = 60 and 4000 ÷ 100 = 40.

This table is very helpful for showing multiplication by powers of 10. Going down a row has the effect of multiplying by 10, while going down two rows produces a multiplication by 100. Similarly it demonstrates very nicely the fact that multiplication and division are inverse operations. Going up a row causes division by 10, and two rows division by 100.

The table can easily be extended to show decimals by adding a row above the numbers 1 to 9:

0.1	*0.2*	*0.3*	*0.4*	*0.5*	*0.6*	*0.7*	*0.8*	*0.9*
1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*

So 0.6 x 10 is 6 and 4 ÷ 10 = 0.4

Give children random multiplication diagrams which use multiplying by 10:

Ask them to find the missing numbers:

x	2		7
	40		
10		50	

Children most often come across multiplication tables in the conventional square with multiplication by 2 at the top and multiplication by 9 at the bottom. It can be more interesting to be given smaller multiplication facts in a random order and so more motivating. Here children have to deduce that the first vertical column is part of the x 2 multiplication table, so the square below the 40 must be found by 2 x 10 and so 20 must be entered. They can deduce that the middle table must be x 5 so as to get the 50.

Multiplying and dividing by single-digit numbers and multiplying by two-digit numbers

Once children are familiar with some multiplication facts, they can use these facts to work out others. One strategy that can be used is writing one of the numbers as the sum of two others about which more is known: 6 x 7 = 6 x (2 + 5) = 6 x 2 + 6 x 5. Another strategy is making use of factors, so 7 x 6 is seen as 7 x 3 x 2. A third strategy is to use a method of doubling, so that 9 x 8 is seen as 9 x 2 x 2 x 2. Further examples of this strategy can be found on pages 50 and 51.

Expectations

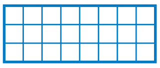
Year 2	9 x 2
	5 x 4
	18 ÷ 2
	16 ÷ 4
Year 3	7 x 3
	4 x 8
	35 ÷ 5
	24 ÷ 3
	23 x 2
	46 ÷ 2
Year 4	13 x 9
	32 x 3
	36 ÷ 4
	93 ÷ 3
Year 5	428 x 2
	154 ÷ 2
	47 x 5
	3.1 x 7
Year 6	13 x 50
	14 x 15
	153 ÷ 51
	8.6 x 6
	2.9 x 9
	45.9 ÷ 9

Note

Some of these examples are not easy. Many children will find it helpful to make jottings while attempting such mental calculations.

Activities

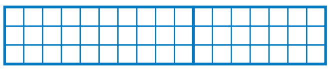
Use an area model for simple multiplication facts. For example, illustrate 8 x 3 as:



How many rows? How many columns? How many squares?

Encourage the children to visualise other products in a similar way.

Extend this model to larger numbers, such as 17 x 3: split the 17 into 10 + 7 and use 10 x 3 + 7 x 3.

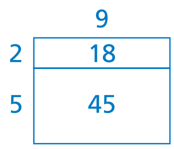


How many rows? How many columns? How many squares?

The rectangles provide a good visual model for multiplication: the areas can be found by repeated addition (in the case of the first example, 8 + 8 + 8), but children should then commit 3 x 8 to memory and know that it is the same as 8 x 3.

Use multiplication facts that children know in order to work out others.

For example, knowing 9 x 2 and 9 x 5, work out 9 x 7:



In area models such as these, the use of repeated addition is discouraged, the focus being on the separate multiplication facts. The diagram acts as a reminder of the known facts, which can be entered in the rectangles, and the way they are added in order to find the answer.

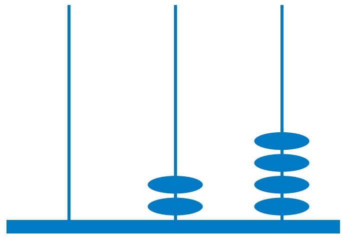
Use base 10 material to model multiplication.

Ask a child to put out rods to represent, say, 24.

Then ask for two more groups of 24 to be added, making three groups altogether. Make sure the child knows that as soon as there are 10 'ones' they are exchanged for a 'ten' or a 'long'.

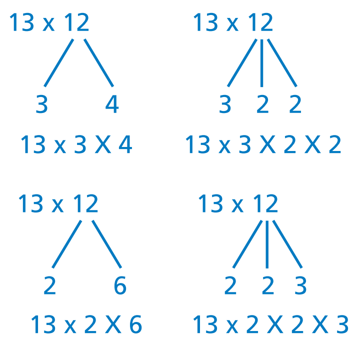
Record the result as 24 x 3.

Do the same, using a spike abacus. This time, as soon as there are 10 beads on a spike, they are removed and replaced by one bead on the spike to the left.



Base ten material is an excellent model of the way in which we group numbers in tens. It lends itself to the concept of multiplication by 10 as each time a 'one' becomes a '10' and a '10' becomes a '100'. Spike abacuses (or abaci) focus on place value, with multiplication by 10 resulting in the 10 beads on one spike being replaced by a single bead on the spike to the left.

Use factors to help with certain calculations. For example, do 13 x 12 by factorising 12 as 3 x 4 or 6 x 2:



Discuss which factors children prefer to use.

These diagrams can help children to keep track of the separate products when they split a number into its factors.

Activities

Display a multiplication tables chart and see how many multiplication facts the children know already. Shade these in on the chart.

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50

… and so on to 100.

Get the children to look at the rows and columns. Make sure they know that there is, for example, a row that starts 2, 4, 6, 8, and a column that has the same numbers. See how far they can continue the even numbers.

Point out the connection between the numbers in the 4 times column and the 2 times column. Ask them about the 8 times column.

Look at the 5 times column. Discuss the fact that these numbers end in a 5 or a 0, and that they are half the corresponding numbers in the 10 times column. Look at the patterns of the 9 times table.

Ask which numbers appear most often in the table. Why?

Ask which numbers less than 100 do not appear in the table. Why?

Discussion of the patterns to be found in such a multiplication table can help children to commit multiplication facts to memory. They might think that every number appears somewhere in the table, so it is helpful for them to think why numbers such as 11, 19 and so on do not appear (all the prime numbers after 7 are missing).

Get the children to practise multiplying by multiples of 10 by putting a set of examples like these on the board:

5 x 10	12 x 10	8 x 20	13 x 20
18 x 20	35 x 30	120 x 20	125 x 30

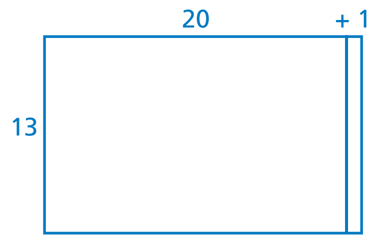
Ask individual children to give the answer to any chosen question. Invite them to make up the most 'difficult' question example that they can do in their head.

Then put examples like these on the board (with the earlier ones still visible) and ask the children to suggest answers:

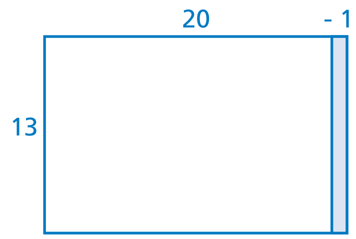
13 x 19	13 x 21	18 x 21
18 x 19	160 x 19	

Discuss how they worked them out.

For 21 x 13, for example, they can think of 20 x 13 and add another 13. A rectangle like this might help them to visualise it:



and for 19 x 13:



The rectangles can help children to see how multiplication by 21, for example, can be carried out by multiplying by 20 + 1, and multiplication by 19 can be seen as multiplying by 20 – 1.

Write this multiplication on the board: 35 x 14.

Explain to the class that this might seem too difficult to carry out mentally, but that there are ways of making it easier to do. Challenge them to find a method that works for them.

They might choose to use the factors of 14, and say:

35 x 14 = 35 x 2 x 7

so they work out 35 x 2 to give 70 and then multiply 70 by 7.

or they might partition the 35 into 30 + 5 and say:

35 x 14 = (30 x 14) + (5 x 14)

Activities

Give them some other examples which can be worked either by factors or by partitioning, such as:

27 x 12 or 48 x 15

Discuss the special case of multiplying by 25 and 50, which is easily done by multiplying by 100 and dividing by 4 or 2 respectively.

The use of factors often makes a multiplication easier to carry out.

Doubling and halving

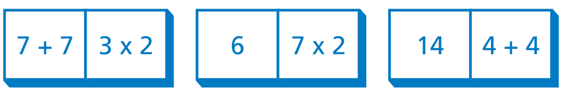
The ability to double numbers is a fundamental tool for multiplication. Historically, all multiplication was calculated by a process of doubling and adding. Most people find doubles the easiest multiplication facts to remember, and they can be used to simplify other calculations. Sometimes it can be helpful to halve one of the numbers in a product and double the other.

Expectations

Year 1	7 + 7 is double 7
Year 2	7 + 7 = 7 x 2
	Half of 14 is 7
	Half of 30 is 15
Year 3	18 + 18 is double 18
	Half of 18 is 9
	60 x 2 is double 60
	Half of 120 is 60
	Half of 900 is 450
	Half of 36 is 18
Year 4	14 x 5 = 14 x 10 ÷ 2
	12 x 20 = 12 x 2 x 10
	60 x 4 = 60 x 2 x 2
Year 5	36 x 50 = 36 x 100 ÷ 2
	Half of 960 = 480
	Quarter of 64 = Half of half of 64
	15 x 6 = 30 x 3
Year 6	34 x 4 = 34 x 2 x 2
	26 x 8 = 26 x 2 x 2 x 2
	20% of £15 = 10% of £15 x 2
	36 x 25 = 36 x 100 ÷ 4 = (36 ÷ 4) x 100
	1.6 ÷ 2 = 0.8

Activities

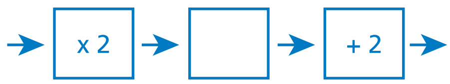
Play 'Doubles dominoes' with a group of children: this needs a set of dominoes in which, for example, 7 x 2, 2 x 7, 7 + 7 and 14 can be matched:



Watch to see which facts the children can recall quickly.

The children can make the dominoes first and then use them to check which multiplication facts they know. They can make a note of the ones for which they did not have instant recall.

Use the 'doubling' and 'halving' function machines:

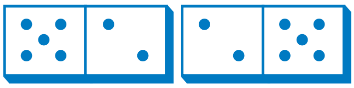


Ask one child to choose a number and another to choose whether to use the 'doubling' or the 'halving' machine. Then ask a third child to say how the number is transformed by the machine.

The function 'machines' focus on the operation x 2 or ÷ 2 so children can give each other lots of quick practice.

Activities

Use 'doubles' from a set of dominoes to make patterns like this:



and count the dots (14).

What other dominoes can be placed in this way to make a total of 14 dots? 18 dots? 20 dots, and so on?

This provides opportunity for lots of short, quick practice.

Doubling and halving number chains.

Ask someone in the class to choose a number. The rule that they are going to use is:

'If the number is even, halve it; if it is odd, add 1 and halve it.'

Go round the class and invite an answer. Then the rule is applied to the answer and a new number is generated and so on, until they get to 1. Write all the numbers in the chain on the board or an overhead projector. For example:

14 → 7 → 4 → 2 → 1

Then ask for a new starting number and continue as before.

Number chains can be quite intriguing as it is usually not possible to guess what will happen. As more and more starting numbers are chosen, the chains can build up to a complex pattern. For example, the starting number 8 joins the chain above at 4; the starting number 13 joins the chain at 7. A starting number of 23, for example, goes to 12, then 6, then 3, then joins the chain at 2.

Ask the children to halve a two-digit number such as 56. Discuss the ways in which they might work it out. Show the children that, unless they can do it instantly, it will probably be better to partition it as 50 + 6 and to work out $\frac{1}{2}$ of 50 and $\frac{1}{2}$ of 6 and to add these together.

Ask a child to suggest an even two-digit number and challenge the other children to find a way of halving it. Some children might be able to respond to halving an odd number (7, say), by saying that it is 3 and a half.

Children should be familiar with halves of multiples of 10 up to 100, so that they can say instantly that half of 80 is 40 and so on. Then larger numbers can be partitioned and the halving facts they know applied separately. Similarly if they are familiar with the halves of multiples of 10 above 100, they can partition three-digit numbers in order to halve them. So they could halve 364, for example, by saying that it is 150 + 30 + 2.

When finding 20% of an amount, say £5.40, discuss with the children that it is easier to find 10% of it first and then double. 10% of £5.40 is £0.54, so 20% of £5.40 is £1.08.

Ask the children how they would find 5% of £5.40. Then get them to work out 15% of £5.40.

This activity assumes that children are familiar with the idea of finding 10% of an amount, that is they know that 10% means 10 out of every hundred or one tenth. Then encourage them to use a range of methods for working out other percentages. For example, they might find 15% of £5.40 by finding 10% then halving that to find 5% and adding the two together. Or, having found 5% they might multiply that result by 3. They could work out $17\frac{1}{2}$% by finding 10%, 5% and $2\frac{1}{2}$% and adding all three together.

Challenge the children to find some 'real life' examples that might result in the need to multiply, say, 12 by 15, or to divide, say, 45 by 3. Then ask them to explain how they would do the calculation.

Getting children to find the 'question' rather than the 'answer' can be revealing about their understanding. So asking them to find everyday situations that would result in a particular multiplication or division gives you the chance to show if they understand these processes. It also helps to emphasise that mathematics has a practical use.

Fractions, decimals and percentages

Children need an understanding of how fractions, decimals and percentages relate to each other. For example, if they know that $^1/_2$, 0.5 and 50% are all ways of representing the same part of a whole, then the calculations

$^1/_2$ x 40
40 x 0.5
50% of £40

can be seen as different versions of the same calculation. Sometimes it might be easier to work with fractions, sometimes with decimals and sometimes with percentages.

There are strong links between this section and the earlier section 'Multiplying and dividing by multiples of 10'.

Expectations

Year 2	Find half of 8
	Find half of 30
Year 3	Find one third of 18
	Find one tenth of 20
	Find one fifth of 15
Year 4	Find half of 9, giving the answer as $4^1/_2$
	Know that 0.7 is $^7/_{10}$
	Know that 0.5 is $^1/_2$
	Know that 6.25 is $6^1/_4$
	Find $^1/_2$ of 36
	Find $^1/_2$ of 150
	Find $^1/_2$ of £21.60
Year 5	Know that $^{27}/_{100}$ = 0.27
	Know that $^{75}/_{100}$ is $^3/_4$ or 0.75
	Know that 3 hundredths is $^3/_{100}$ or 0.03
	Find $^1/_7$ of 35
	Find $^1/_2$ of 920
	Find $^1/_2$ of £71.30
	Know that 10% = 0.1 = $^1/_{10}$
	Know 25% = 0.25 = $^1/_4$
	Find 25% of £100
	Find 70% of 100cm
Year 6	Know that 0.007 is $^7/_{1000}$
	Know that 0.27 is $^{27}/_{100}$
	0.1 x 26
	0.01 x 17
	7 x 8.6
	Know that 43% is 0.43 or $^{43}/_{100}$
	Find 25% of £360
	Find $17^1/_2$% of £5250

Activities

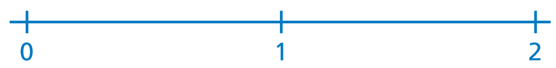
Write a sum of money on the board, for example £24.

Ask children, in turn, to tell you what half of £24 is, then $\frac{1}{3}$, then $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{8}$ and $\frac{1}{12}$.

Then give fractions such as $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$, $\frac{3}{8}$, $\frac{7}{12}$ and ask how they could calculate these fractions of £24.

In answering questions such as these, children will use the basic strategy of using what they already know to work out related facts. In this case they will need to know how to find fractions such as $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$ of an amount and to use this to find other fractions. For example, knowing that $\frac{1}{3}$ of £24 is £8 they can say $\frac{2}{3}$ is twice as much, or £16. Similarly, knowing that $\frac{1}{8}$ of £24 is £3, then $\frac{3}{8}$ is three times as much.

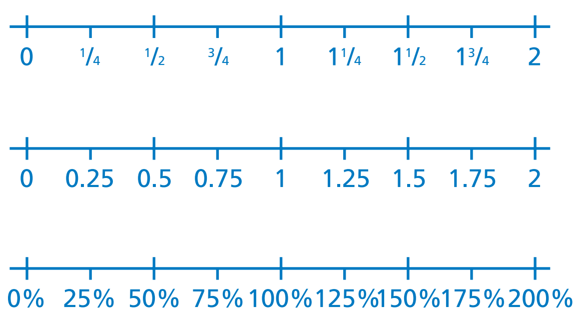
Draw a number line on the board, marking on it the points 0, 1 and 2:



Invite children to show where the fractions $\frac{1}{2}$, $\frac{3}{4}$, $1\frac{1}{2}$, $\frac{1}{4}$, $1\frac{3}{4}$, and $1\frac{1}{4}$ fit on the line. Ask what other fractions between 0 and 2 they could add to the line.

When they are familiar with fractions, draw a new line under the first one and ask for the decimals 0.5, 1.5, 0.25, 1.25, 1.75, 0.75 to be placed on this line. Repeat with a line for percentages from 0% to 200%.

Then discuss the three lines:



Choose any number and ask children to call out the number above it or below it. Discuss the equivalence of, for example, 1/4, 0.25 and 25%.

The use of number lines for including fractions between the whole numbers is very helpful in trying to establish the idea that fractions are numbers and in moving children away form the part-whole model of fractions. The first part of the activity requires children to think about the relative sizes of fractions. Separate number lines with, for example, halves, quarters and eighths placed one under the other help to establish the idea of equivalent fractions. Gradually they can be built up to include more fractions, up to, say, twelfths. And, as in the example above, they can also be used to demonstrate the equivalence between fractions, decimals and percentages.

Put a percentage example on the board, say 25% of £60.

Discuss different ways of interpreting the question, such as $\frac{25}{100}$ of £60 or $\frac{1}{4}$ of £60.

Children might choose to calculate $\frac{1}{4}$ of £60, saying $\frac{1}{2}$ of £60 is £30 and $\frac{1}{2}$ of £30 is £15, or they might say:

10% of £60 is £6, so 5% of £60 is £3 and 20% of £60 is £12, so 25% of £60 is £12 + £3.

Then ask them to find, for example, $17\frac{1}{2}$% of £60. Knowing that 5% of £60 is £3 they can work out that $2\frac{1}{2}$% of £60 is £1.50. Then they can find the total by adding 10% + 5% + $2\frac{1}{2}$%.

Invite the children to suggest other examples.

It would help if children have access to actual examples that they might have seen in local shops or in newspapers, in order to provide a more motivating context. It is important that children understand the basic fact that % means 'out of a hundred' or 'per hundred', rather than to learn any rules about working with percentages. The examples they are likely to meet at this stage can all be worked out using just this basic knowledge.

Part 5
Using calculators

The place of the calculator in primary mathematics

Calculators are powerful tools and, as with all tools, it is necessary to learn how to use them properly. They are not an appropriate tool for calculations that can more quickly and reliably be carried out mentally or by written methods. However, used well, calculators can help children to learn about numbers and the number system. Children need to learn to judge when it is and when it is not sensible to use a calculator. It is also important for them to learn how to use a calculator efficiently, including how to use calculator functions such as the memory key and the constant function.

This guidance clarifies the role of the calculator in primary schools by making clear that:

- **calculator use in primary classrooms should be carefully controlled by the teacher to ensure that children do not use calculators as a prop for calculations which can and should be done mentally or with pencil and paper;**
- **calculators should not be used as a calculating aid until years 5 and 6 when children are able to add and subtract any pair of two-digit numbers in their head.**

Calculators are powerful tools for supporting the development of mathematical concepts and understanding of the number system. Although much of this work would take place during key stage 2, there may be times when key stage 1 teachers could use effectively an OHP calculator when working with the whole class to secure the teaching of a particular concept.

In the 'Introduction' to the *Framework for teaching mathematics* there is a good section on the role of calculators. Teachers might wish to cross-reference to this section of the framework and also to the section of examples for children in years 5 and 6.

Calculator usage

Calculators can be used as a teaching aid to promote mental calculation and the development of mental strategies. The use of transparent calculators designed for the overhead projector is particularly effective for generating discussion, either in a whole class or group setting.

Calculators can stimulate problem solving in mathematics and provide opportunities for developing mathematical concepts that would otherwise be inaccessible.

The calculator, though, however it is used, does not do the strategic thinking for the user. It is important, before any step in a calculation is carried out, to decide what operation is appropriate. Equally important is to ask whether the result makes sense and how to interpret the display. Whenever children use a calculator, they should be encouraged to ask themselves questions such as:

What numbers do I need to enter?
In what order do I key in the numbers?
Do I have to add, subtract, multiply or divide these numbers?
Does the result make sense?
How do I interpret the display?

A calculator allows children to work with real data, without the need for teachers to simplify the numbers being used artificially. In this way, the calculator can open up a range of real problems that might arise within mathematics or from other aspects of the curriculum, such as science or geography.

Hence, the three important uses of the calculator are:

1 using a calculator for real data and for awkward numbers;

2 using a calculator to explore and gain understanding of mathematical ideas and patterns;

3 using a calculator to promote mental calculation skills.

These uses are discussed in more detail in the following pages.

Using a calculator for real data and for awkward numbers

There are many tasks that are carried out most successfully with the aid of a calculator. Such tasks often arise out of enquiry into problems of genuine interest to the children but which may involve calculations that are too awkward to deal with mentally. It is important for enquiries of this type to be made because they encourage children to see that mathematics is useful and relevant to real problems and they may help them to begin to understand the importance of the subject. A mathematics programme in which all the numbers or measurements ever met are small enough and simple enough to be dealt with mentally creates a limited and artificial view of mathematics and fails to give children an understanding of how mathematics is used in the world outside school.

An important feature of providing real contexts for children to work with is that they provide a need for a decision to be made as to which mathematical operations are appropriate for the task. Examples in which children are directed, for example, to add or divide two given numbers certainly offer opportunities for practice, but they do not provide children with the skills to make decisions when confronted with real problems. When the development of these analytical skills is the main objective of a lesson, it is appropriate to use a calculator so that children are less likely to be distracted by problems in carrying out calculations. The calculator is just a tool: it makes no decisions. Its operator needs to analyse the task and decide what information is relevant and how it should be used.

Not only does the calculator make no decisions, it also will not interpret the display that arises from a calculation. If a child needs, say, to find how many coaches, each with a capacity of 54 people, will be needed to take 347 children on an outing, he or she may begin by using a calculator to divide 347 by 54. The display 6.4259259 is not, as it stands, a real solution to the problem and has to be interpreted. The child needs to know that the number of coaches has to be a whole number, in this case the next largest one, 7. If he or she also wants to find the number of spare places, further work has to be done. The child might calculate 6 x 54 to get 324 and see that there are 23 seats too few. Multiplying 7 x 54, giving 378, he or she can see that there are 31 spare seats.

The examples given below are suitable for years 5 and 6, both in mathematics and in other subjects. They include cases that require children to make decisions as to which operations they need to use, which calculator keys they need to press and in which order to press them. They require children to interpret the resulting display on their calculator. Notes about the activities appear in blue.

The least and greatest heights of girls in a class are 118.7cm and 161.2cm. Find the range of heights of the girls in the class.

This is a realistic handling data activity where the awkwardness of the data values warrants the use of the calculator. The activity can be extended to other contexts in mathematics itself, or in other areas of the curriculum, such as geography or science, where the range of values is of interest.

Collect some information on comparative prices for large and small amounts of any items. Ask, for example: 'Which is the better buy: 567g of ketchup at 89p or 340g at 49p?'

As the price of many goods is given as, for example, a cost per 100g, comparisons are now easier to make without calculation. But tasks such as these provide opportunity to explore ideas of ratio.

An old model village has houses which were built to a scale of 1 in 17. One model shop is 47cm high. What would be its real size? A model is made of a school that is really 11.85m high. How high would the model be?

Some early model villages were built with strange scales, in this case 1 in 17. Children will need to decide whether they have to multiply or divide by 17 before they can use their calculator to carry out the appropriate operation. This kind of activity can be extended to map scales.

250 kilometres is approximately the same as 155.3 miles. How many kilometres are there to 60 miles?

Children could think how they would use the information to find the number of kilometres in 1 mile (ie by dividing 250 by 155.30 and then multiplying by 60 to find the distance in kilometres).

To change a Celsius temperature to a Fahrenheit temperature, multiply by 1.8 and add 32. To change a Fahrenheit temperature to a Celsius one, subtract 32 and divide by 1.8. Using these rules, what is the Celsius equivalent of 83°F? What is the Fahrenheit equivalent of 15°C?

The number 1.8 arises from comparing the 180 degrees between the freezing point and boiling point of water on the Fahrenheit scale with the 100 degrees between the same two temperatures on the Celsius scale.

A machine makes 752 300 drawing pins every day. These are packed in boxes of 125 pins. How many boxes can be filled in one day?

While it may be possible for some children to carry out a calculation like this mentally, for others the numbers will be too large. In either case, the first task is to decide what sort of calculation to perform – in this case, 752 300 has to be divided by 125.

Petrol costs 65.9 pence a litre. My petrol tank holds 41.5 litres. How much does it cost to fill my petrol tank? My car can travel about 340 miles on a tank full of petrol. What is the cost per mile?

The first decision has to be to multiply 65.9 or 0.659 by 41.5 to get the total cost. Then the calculator display has to be interpreted. Knowing the cost for 340 miles, children will need to divide to find the cost per mile.

The planet Mars is 0.11 times heavier than Earth. Jupiter is 317.8 times heavier than Earth. How many times heavier is Jupiter than Mars?

To find how many times heavier Jupiter is than Mars, children will need to find how many times 317.8 is bigger than 0.11. Some children tend to assume all comparisons are made by subtraction, so they will need to think carefully about the importance of the word 'times' in the question. They need to find, in effect, how many times 0.11 will divide into 317.8. They will also need to interpret the display.

In 1955 it was decided that there were 31 556 926 seconds in that year. How many minutes were there in the year? How many hours in the year?

Children need to learn to take care in entering large numbers. After dividing by 60, they will need to interpret the calculator display to give an answer in minutes. At what stage should the interpretation of the display be made when calculating the number of hours in the year?

A computer printer can print 346 characters per second. How many characters are there on this page? How long will it take to print the page?

Children will need to decide how to arrive at the number of characters in a given piece of writing. Then the calculator will help them to arrive at an estimation of how long it will take to print.

Investigate the sequence: 1 1 2 3 5 8 13 21 … Each number is the sum of the two previous numbers. Divide each number by the one before it: 1 ÷ 1, 2 ÷ 1, 3 ÷ 2, 5 ÷ 3… What do you notice if you keep going?

Some children may be familiar with the Fibonacci sequence; it has many interesting properties. In this case, as children carry out the successive divisions, their calculator display will eventually show numbers that always begin 1.618 … In fact, the ratio of adjacent terms in this sequence gets closer and closer to what is known as the Golden Ratio, which could be the beginning of further investigation.

The two examples that follow can be used as extension activities for more able children. The mathematics addressed in each of these activities lies outside the mathematics content specified in the 'Framework'. They are typical of calculator activities that a teacher could devise for children working beyond level 5 in mathematics.

The distance from Brussels to Milan is approximately 785 miles. The air flight took one hour 35 minutes. What was the average speed of the aeroplane?

Knowing that the journey time is 95 minutes, children will need to divide 785 miles by 95 and then multiply the result by 60 to find the distance travelled per hour.

Six similar tubes of sweets were found to contain 46, 49, 47, 45, 51 and 49 sweets respectively. What was the mean number of sweets in the tubes? How many tubes of sweets can be filled from a container that has 5000 sweets?

The mean number of sweets per tube can be calculated in order to get an idea of how many would be filled from the large container. Alternatively, they could use the smallest number and the largest number of sweets to see what effect that has on the number of tubes that can be filled.

Using a calculator to explore and gain understanding of mathematical ideas and patterns

Calculators can help children to enhance their understanding of some fundamental mathematical ideas, such as negative numbers or place value. Calculators can also provide opportunities to explore mathematical ideas that would otherwise be inaccessible, but which have intrinsic value. These include divisibility, recurring and terminating decimals and the effect of multiplying by numbers between 0 and 1. These ideas lend themselves readily to classroom discussion. Transparent calculators used with an overhead projector are invaluable in promoting whole class discussion, as everyone can the see the display and the effect of entering or changing any number or operation.

Enter '10 – 1 ='. Keep pressing the = button and count down with the display. Go past 0 and see what happens. Enter '103 – 10 =' and keep pressing the = button. Enter '57 – 5 =' and keep pressing the = button.

This activity works well with an overhead projector so all children can see the display. The teacher can encourage prediction and reasoning by asking 'What will come next?'.

Exploring sequences

The teacher sets the constant function on an OHP calculator to create sequences, and then asks questions about each sequence. For example: 3 + 5 = = = …

– How does the units digit in the display change?

– What will the next number be?

– What number will appear in the display after ten presses of the = key?
  4 + 3 = , = …

– How do the units digits in the display change?

– Why does this happen?

– Is it possible to 'hit' the number 20, 200 …?

– After how many presses will you pass 100?

The calculator provides a quick way of creating sequences and these can be used to explore a range of numerical properties.

Place value

One child enters a three-digit number on a calculator, for example 374. The second child suggests a change to one of the digits, for example 354, and challenges the first child to change the number to 354 with just one subtraction (– 20). He then challenges her to make the new display into, say, 854 with just one addition (+ 500). They could set themselves a target number, such as 888 or 654, at which they will aim.

This game helps to reinforce the idea of place value. The children can announce, in advance, what operation they will use, in which case the calculator provides a check.

Largest answer

Use the digits 1, 2, 3, 4 and 5 in any arrangement and one x sign. You could have 241 x 35 or 123 x 54, etc. What is the largest answer you can get?

This activity helps to focus attention on place value. Having to think about getting the largest product encourages children to think about where to put each number. Children can choose other sets of numbers, for example 5, 6, 7 and 8 .

Matching divisions

Choose any number from this set: 1, 2, 3, 5, 8, 4, 6 … and another from this set: 3, 6, 8, 12,10, 4, 6, 16, 32, 20, 15 … Divide the first number by the second. Find some pairs that give the same answer. Make up some more pairs that will give the same answer.

This activity helps children to work with decimal equivalents of fractions.

Multiplying by a number close to one

Choose any two- or three-digit number. Use the calculator to multiply it by 1.1. Try it with different starting numbers just greater than one. What can you say about the answers? Now try multiplying by 0.9. What happens now?

In this activity the calculator readily shows the effect of multiplying by numbers just greater than or less than one. During discussion, children can explore the reason why this is so.

Matching multiplications

Give the children two sets of numbers, such as A (7, 15, 23, 34, 47, 53, 137) and B (782, 645, 371, 795, 1219, 959, 705, 345, 1802, 1081, 510, 1598). The first child chooses two numbers from set A that they think will multiply to a specified number in set B. For example, one child might think, incorrectly, that 15 x 23 = 645. Another child checks with a calculator. If the answer is correct, that number is coloured in by the first child. Then the second child has a turn and the game continues until one of the children has four numbers in set B coloured in.

This game encourages children to consider what happens when two numbers are multiplied, for example if one of the numbers ends in a 5, the product will also end in a 5, or if one number ends in a 2, the product must be even, or if one number ends in a 6, the product must end in a number that is the unit's digit in the 6 times table, ie 6, 2, 8, 4 or 0. Such considerations help to improve mental calculation strategies.

Prime factors

The number 74 865 can be written as the product of prime numbers. What are they?

Children will need to use what they know about number in order to begin this activity. For example, they may decide to divide the number by 5 first.

The next activity is an extension activity for more able children. The children will need to know how to use the constant function on the particular calculator being used.

A. Explore sequences such as:

23 + (-2) =, =, =, ...

1.75 + (-0.5) =, =, =, ...

Such sequences will give the children experience of calculating with negative numbers. The sequence below will give children further experience with halving, and will also show the continued effect of multiplying by a number between 0 and 1.

B. Explore sequences such as:

9 x 0.5 =, =,=, ...

Using a calculator to promote mental calculation skills

There are many opportunities to make use of a calculator to develop mental calculation skills. Many of these take the form of challenges between two children, one of whom works mentally while the other attempts to arrive at the solution more quickly with a calculator.

This sort of activity encourages the use of mental methods; the challenge is to beat the calculator.

Near doubles

One child has a set of cards, each with an addition of two numbers that are close, eg 35 + 36, 48 + 49, 125 + 126. The other two children have to try to be the first to get the answer, one by mental calculation and the other with a calculator.

This is a case where the calculator and mental calculation are compared for speed. Children who know the doubles should find they can get to the answer mentally before the one who uses the calculator.

Beat the calculator

This is a game for two players. They work with a set of cards such as: 3 x 8, 21 x 7, 5 x 9 and 13 x 3 that are face down. Player A turns over a card and challenges player B to give the answer before player A can produce the answer on the calculator. Whoever is first wins the card.

Games like these encourage children to increase their speed of calculation and the number of facts which they can recall rapidly.

Counting forwards and backwards

Set up two OHP calculators to act as simultaneous counters. For example, set calculator A to count forward in twos, starting at 0 (0 + 2 = = =), and calculator B to count backwards in threes from 100 (100 – 3 = = =). Ask questions such as: 'What number will appear next on each calculator?' 'What happens to the sum of the pair of numbers that appear at any stage and to the difference?' 'Will the same numbers ever appear on both calculators at the same time? If so, which ones?'

The role of the calculator in this activity is to provide a context for asking a range of numerical questions that provoke mental calculation and stimulate class discussion.

The 100 complements game

Two children take turns to choose two numbers from this array that they think add to 100.

50	65	26	30	19	85	10	96
35	39	36	18	31	52	73	46
55	28	37	77	22	12	17	20
51	81	59	1	69	70	9	25
72	90	78	88	48	5	82	39
11	75	64	49	50	63	38	99
80	27	45	61	23	74	41	54
91	4	15	62	95	83	15	89

After each player chooses two numbers, the other player checks the answer with a calculator. If the answer is correct, the player places a counter on the two numbers and they can not be used again. The player with the most counters when all numbers have been paired wins.

This game provides motivation for mental calculation. However, it is important that the suggested numbers should be checked. The calculator acts as a surrogate teacher; the child with a calculator will be motivated to see that the other player does not make an error.

Operations

Set up two calculators with two different functions. For example, one child can be given a calculator that has been programmed to perform + 2 (2 + =) and another with one programmed to multiply by 2 (2 x =). The class suggests a number for the first child to key into their calculator. The child shows this number to the second child who keys it into the other calculator and gives the class the result. They then have to work out what the calculators have been programmed to do. The class can then predict the outcome for given inputs and the input for given outputs. Other functions can be tried.

The calculator is used to provide the motivation for the class to carry out mental calculation.

Interpreting the display

i) Four books cost £6. What does one book cost?

ii) At a school concert, programmes cost £0.35. How much money is collected when 140 programmes are sold?

Simple questions such as this can be used by the teacher to help develop children's sense of having to interpret the calculator display in terms of the original problem. For (i), the calculator display will show 1.5 which has to be reinterpreted as £1.50; similarly, in (ii), the display of 4.9 has to be reinterpreted as £4.90. Such activities can be varied so that, for example, children have to round up to the nearest integer, as in the example 'How many 12-seater minibuses are needed for a school outing of 70 children and 7 teachers?'

Part 6
Approximating and checking

Approximating

Mathematics is often regarded as a subject in which all answers have to be either right or wrong. However, there are many occasions in everyday life when an approximate answer to a problem is appropriate. For example, the newspaper headline: '750 000 fans welcome home the victorious French football team' clearly does not mean that exactly 750 000 people were there, but rather that someone had made an estimate of the number. Similarly, the value of π is usually taken as 3.14, because that is sufficiently accurate for the purpose, even though it is actually a decimal that never ends. In measurement, the degree of accuracy required depends on the task, so any measurement is an approximation.

It is quite hard for children who have been accustomed to giving answers to understand the idea that it is not always either possible or necessary to do so. The idea of 'suitable for the purpose' needs time to develop.

Most children are probably familiar with the idea of estimating, when used in the context of measurement. The length of the classroom, for example, might be estimated in 'footlengths', making use of previous experience of such measurements. The number of 'footlengths' will be approximate, but sufficiently accurate for the purpose.

Another area in which making an approximation is useful is before embarking on a calculation, as it can help to reduce later mistakes. So, before carrying out, say, 37 x 53, it is useful to know in advance that the answer will be more than 30 x 50 and less than 40 x 60. It may also be helpful to know that a closer approximation is 40 x 50. This strategy involves the idea of rounding a number to the nearest ten, or hundred or whatever is appropriate. The fact that these approximations can be done mentally means that they provide a way of getting a rough answer even when paper and pencil, or a calculator, are not available.

Some activities that help children to understand the ideas involved with approximation are given below:

Ask the children to estimate the number of pencils in a box, or the number of beads in a jar. They can see how good their estimate is by actually counting the objects, but they should not think that their estimate is wrong because the two numbers are not exactly the same. Suggest ways in which they might carry out the estimation.

Ask the children to bring examples of newspaper headlines in which large numbers are involved. Discuss with them whether the numbers are exact or not.

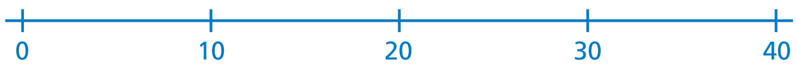
Get a group of children to play ‘Four in a row’. They will need two numbered dice or spinners and a 5 x 5 baseboard containing 25 randomly-placed multiples of 10 such as:

50	30	10	20	80
70	20	90	40	10
80	30	60	10	50
50	20	70	0	90
30	40	10	60	40

The children take turns to throw the dice and, writing the digits in any order, make a two-digit number.

They should round their number to the nearest 10 and then place a counter on that number on the board. The aim of the game is to get four counters in a row.

Use number lines, such as:



Ask the children, in turn, where they would place the numbers 18, 25, 7, 32 and so on. Encourage them to explain how they decided where to place their number.

What strategies do the children use?

Extend the number lines used to include ones such as:

Where would you place 370, 320 …?

Where would you place 2.5, 2.8 …?

Where would you place 1.15, 1.13 …?

Encourage them to explain how they decided where to place each number. Discuss the strategies they used.

Get individual children or pairs of children to play 'Target'.

Give them some digits and ask them to decide how to place them in the boxes so the boxes give an answer as near to the target as they can. For example:

3, 4, 6   ☐☐ x ☐   Target 140

6, 7, 8, 9   ☐☐ x ☐☐   Target 6000

Write some calculations on the board, such as 58 + 86 + 31.

Invite the children to say what they think is an approximate answer. They might find two values between which the answer lies, by saying 'the answer lies between 50 + 80 + 30 and 60 + 90 + 40'. Others might round each number to its nearest multiple of 10, giving, this time, 60 + 90 + 30.

Encourage the children to make up some examples of their own and work on them in a similar way.

You might like to get them to find which of the two methods is closest to the actual answer, and to say why.

Then try other examples, such as 953 – 368, 63 x 87 or 953 ÷ 289. These they could clearly not do mentally, but they can get an approximate answer.

Play the 'Rounding game' with a pair of children. They need a pile of single-digit cards, with no zeros. Children take turns to choose a card and place them to make three two-digit numbers. For example, they might get 73 + 56 + 18:

7 3 + 5 6 + 1 8

One child rounds each two-digit number to the nearest 10 and adds them together, getting:

70 + 60 + 20 = 150

The other child adds the original numbers:

73 + 56 + 18 = 147

The player with the greater number scores the difference (3 in this case).

Checking

Although, in general, it is preferable to get children to stop and think about the calculation before they embark on it, getting a rough idea of the size of the answer, there is also a strong case for encouraging them to think about the reasonableness of their answer when they have arrived at one. Experience suggests, though, that it is quite difficult to ensure that this actually happens.

There are three types of checking that are helpful: rough checking, partial checking and exact checking.

Rough checking

This involves looking at the answer to decide whether it seems reasonable. A check can be made to see if the answer is the right order of magnitude, whether it is far too big or far too small. One might do this as in a quick glance at a supermarket check-out bill – asking whether the total seems reasonable. Rough checks are usually done mentally and use the same procedures as for the approximations made before the calculation, as described earlier.

Example:

Ask children why this bill is obviously wrong:

£3.40 + £6.87 + £5.97 + £3.56. The total was given as £81.63.

How do you know it is wrong? What was the error?

Discuss examples like this one:

The answer to 32 x 27 was given as 288.

How do you know it is wrong? What is an approximate answer?

What is 30 x 30? What do you think the mistake was?

How do you know the answer to 5.6 x 8.3 cannot be 4648?

What do you think the error was?

Partial checking

This is a way of deciding that certain features of the answer suggest that it is wrong. This time the question of whether the answer is reasonable makes use of some basic properties of numbers such as whether the answer is even or odd, whether it is a multiple of 5 or 10. The final digit of the answer can be checked. In the case of multiplication, for example, a units digit of 6 can come from multiplying 6 and 1 or 2 and 3. Divisibility rules can also be useful: if a number has a factor of 9, for example, the sum of its digits is divisible by 9.

Example:

Discuss cases such as :

- a child says that 7 x 8 is 55. Ask them why it cannot possibly end in a 5.
- a child says that 13 x 13 is 167. Why is that not possible?
- a child says that 37 x 3 is 921. Why is that not reasonable?

Exact checking

This usually involves carrying out the calculation in a different way. This might involve turning a subtraction into addition or a division into multiplication. Or it might be possible to do the same calculation using a different method.

Example:

Check 13 + 27 + 35 by doing the addition in a different order.

Check 65 – 38 by adding: does adding 38 to the answer give 65?

Check 144 ÷ 16 by multiplying: does multiplying the answer by 16 give 144?

Check 12 x 15 by doing the equivalent calculation 6 x 30

Checking calculator work

Mistakes can easily be made when keying in numbers into the calculator while carrying out a calculation. This is analogous to the errors that can be made between looking up a number in a telephone directory and then dialling that number. So checking an answer when working with a calculator is every bit as important as checking an answer when working with a paper and pencil method. Some of the most common errors that calculator users make are:

- omitting a digit;
- entering an incorrect digit;
- entering digits in the wrong order;
- entering a number from a list twice;
- entering the wrong operation;
- putting the decimal point in the wrong place.

One way of encouraging children to check their calculator answers is to get them to discuss the error that has been made in examples such as:

14.7 x 2.3 = 338.1;

1001 x 13 = 77;

56.71 – 33.47 = 33.24, etc.

Reinforcement

Encouraging children to estimate, to make approximations and to check their answers is a long-term process and one that has to be regularly addressed. Many teachers would agree that it is not easy to get children to see the importance of these aspects of mathematics, and it is probably best to discuss them frequently and to make the most of any opportunity that arises.

Glossary

Approximate (v) To find a number that is close enough to an answer for the context.

Approximate (n) A number that is close enough for the purpose.

Associative law The description of operations in which the result of a combination of three or more elements does not depend on how the elements are grouped, eg 3 + 4 + 7 can be grouped as (3 + 4) + 7 or as 3 + (4 + 7) and 3 x 4 x 7 can be grouped as (3 x 4) x 7 or 3 x (4 x 7). Division is not associative, as (12 ÷ 4) ÷ 3 ≠ 12 ÷ (4 ÷ 3).

Compensating The process of restoring equality by addition and subtraction in order to make a calculation easier, eg 17 – 9 = 17 – 10 + 1.

Commutative law The description of operations in which the order of the elements does not affect the result, eg 3 + 4 = 4 + 3 and 3 x 4 = 4 x 3. Neither subtraction nor division is commutative, as 4 – 3 ≠ 3 – 4 and 12 ÷ 3 ≠ 3 ÷ 12.

Complement (n) A number that is required to be added to produce a given number, usually 10 or a multiple of 10, eg the complement of 7 to 10 is 3, the complement of 96 to 100 is 4.

Consecutive integers A series of numbers with equal steps of one between them, eg 3, 4, 5 or 11, 12, 13.

Constant key A key on the calculator (usually the = key) such that repeated pressing of it, after entering + or x and a number, adds or multiplies by that number repeatedly, eg 4 + 3 = = = … gives the sequence 7, 10, 13, 16 … ; 4 x 3 = = = … may give the sequence 12, 48, 192 … or 12, 36, 106 … depending on the calculator.

Decimal fraction A fraction in which the denominator is 10 or a power of 10, and in which the digits to the right of the decimal point show the number of tenths, hundredths, etc.

Digit A single symbol that represents a counting number from 0 to 9, eg the number 385 contains the digits 3, 8 and 5.

Distributive law The principle that defines the way that two different operations can be combined, for example:
7 x (20 + 6) = (7 x 20) + (7 x 6), or
3 x (7 – 4) = (3 x 7) – (3 x 4), but
3 + (7 x 4) ≠ (3 + 7) x (3 + 4), and
12 ÷ (4 + 2) ≠ (12 ÷ 4) + (12 ÷ 2).
Multiplication is said to be distributive over addition (or subtraction), but addition is not distributive over multiplication, nor is division distributive over addition or subtraction.

Estimate (n) Use of previous experience of a measurement to make a judgement about a specific measure or number.

Estimate (v) To arrive at a rough answer to a calculation by making suitable approximations for terms in the calculation.

Equivalent Numbers or measures that have the same numerical value.

Factors Numbers that divide exactly into a given number, eg 1, 2, 3, 4, 6 and 12 are factors of 12.

Fibonacci An Italian mathematician (c1170–c1230), whose name is given to the sequence 1,1, 2, 3, 5, 8, 13, 21 … in which each term, after the first two, is the sum of the two previous terms.

Fractions The ratio of two whole numbers; also the result of dividing one integer by another integer, eg $^{3}/_{4}$ = 3:4 = 3 ÷ 4.

Function machine An imaginary device that transforms one number into another by a defined rule.

Hundredths Fractions with the denominator 100; digits in the second place after the decimal point, eg in the number 3.64, the digit 4 represents four hundredths or 4/100.

Inverse operations Operations that, when combined, leave the number on which they operate unchanged, eg multiplication and division are inverse operations: 7 x 3 ÷ 3 = 7 x 1 = 7. Addition and subtraction are inverse operations: 7 + 3 – 3 = 7.

Mean A form of average in which the sum of all the elements is divided by the number of elements, eg the mean of the numbers 3, 7, 8, 5 is found by adding 3 + 7 + 8 + 5 and dividing the result by 4.

Multiple A number that has factors other than 1, eg 12 is a multiple of 2, 3, 4 and 6.

Partitioning The process of splitting a number into component parts, eg the number 359 = 300 + 50 + 9.

Percentage The number of parts per 100; the number of hundredths. For example 1/2 = 50/100 = 50%.

Prime number A number that has no factors other than itself or 1, eg 3 … 11 … 29 … 101 … 1093 … The number 2 is a prime number, but 1 is not.

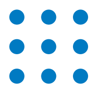
Product The result of multiplying numbers together.

Rectangular array An arrangement of numbers in the form of a rectangular block.

Rounding Finding the number that is closest to a given number that has a more appropriate number of digits, eg 17 rounded to the nearest 10 becomes 20, but rounded to the nearest 100 becomes 0.

Square root A positive number that when multiplied by itself gives the original number, eg the square root of 25 is 5($\sqrt{25}=5$).

Square number A number that results from multiplying one number by itself; a number that can be represented by a square, eg 9 is a square number because 3 x 3 = 9; 9 can be represented as:



Test of divisibility A check to see whether one number divides exactly into another number.

Tenths Fractions with the denominator 10; digits in the first place after the decimal point, eg in the number 3.47, the digit 4 represents four tenths or 4/10.

About this publication

Who's it for?	Teachers, mathematics and assessment coordinators and headteachers in primary schools, LEA mathematics advisers, INSET providers and heads of mathematics in secondary schools.
What's it about?	This booklet offers guidance to teachers on teaching effective mental strategies for calculation and makes clear the expectations concerning the use of calculators.
What's it for?	It is designed to assist teachers in their planning by listing the number facts that children are expected to recall rapidly and giving year-by-year expectations of a range of calculations that children should be able to do mentally. It lists strategies that might be introduced to children and suggests a range of activities for use in the classroom.

This publication has been sent to:
All primary schools covering key stages 1 and 2

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