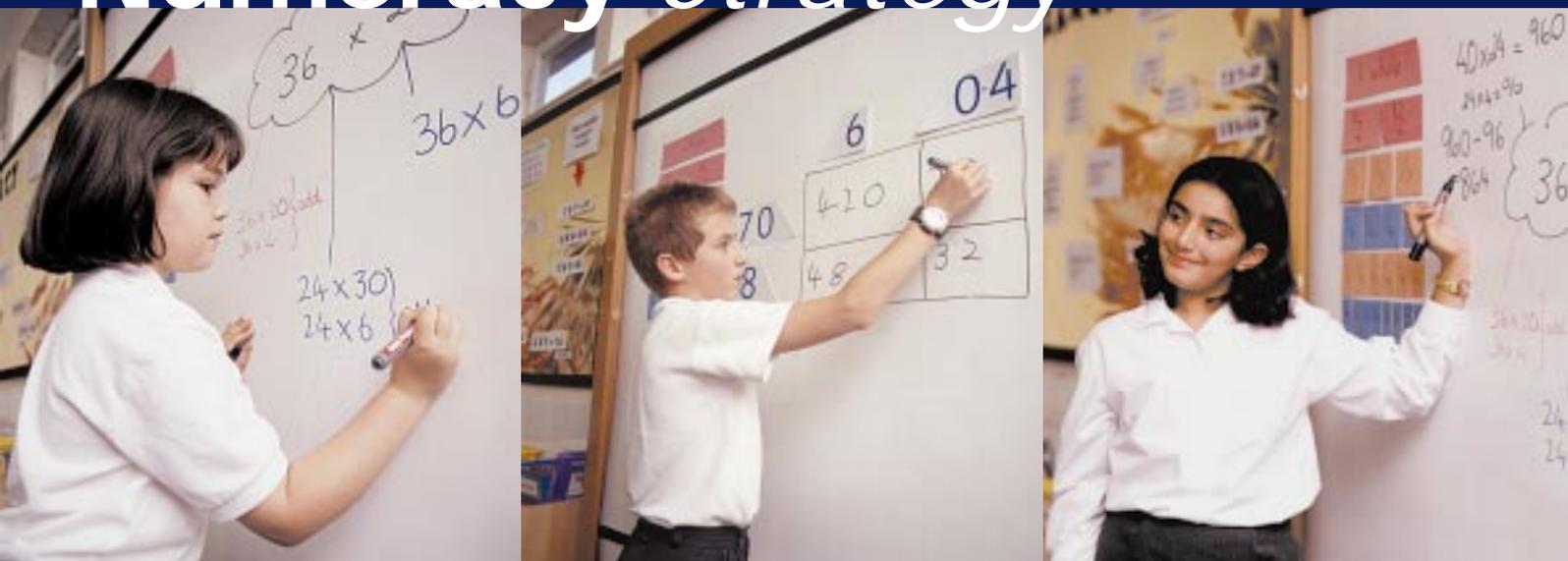


The National Numeracy Strategy



Teaching written calculations
guidance for teachers at key stages 1 and 2

First published in 1999

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Qualifications and Curriculum Authority
29 Bolton Street
London
W1Y 7PD
www.qca.org.uk/

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Introduction

Number is central to the learning of mathematics for children in classes from reception to year 6. During this period, children become familiar with the four operations of addition, subtraction, multiplication and division and learn to apply them to a range of problems arising out of real-life situations. Many children will experience the fascination that working with numbers can bring.

It is important to lay firm foundations in mathematics and to build on these in a systematic manner. The revised national curriculum for the year 2000 and the *Framework for teaching mathematics* make it clear that in the early years of primary school the main emphasis should be on developing skills of mental calculation. The development of formal methods of recording and performing calculations should follow from a firm grounding in learning, understanding and using a range of mental calculation strategies. This is not to say that written recording should not take place in the early years; on the contrary, recording of one form or another should take place regularly and is an essential part of learning and understanding. Each teacher has the responsibility to ensure that learning is reinforced and that children move progressively from informal to more formal methods of recording when they are capable of understanding the underlying mathematical processes.

The purpose of this booklet

The purpose of this booklet is to outline the development of written methods and to give guidance to teachers on how best to achieve this. It deals with when and how to introduce written methods and how these link with mental strategies.

The reasons for using written methods include:

- to assist in a mental calculation by writing down some of the numbers involved;
- to clarify a mental procedure for the writer;
- to help to communicate solutions and methods with other readers;
- to provide a record of work done for themselves, teachers and others;
- to work out calculations which are too difficult to be done mentally;
- to develop, refine and use a set of rules for correct and efficient calculations.

It is well known that some children experience great difficulty in understanding formal methods of written calculation and applying them correctly in a given situation. However, as the *Framework for teaching mathematics* makes clear, it should be the aim that by the end of year 6 all children will understand, and use successfully, conventional standard written methods to carry out and record calculations they cannot do 'in their head'. On the way, children will need to use an expanded layout, but should be encouraged to work towards the most compact form while staying with a level of sophistication that they can understand. It is essential to build on what children know, understand and can do. If more succinct and efficient abstract methods are imposed on children without ensuring their understanding, the result, all too often,

is that they lose even their more primitive methods and become disabled by the ‘teaching’ process. The teaching and learning of written methods should develop from mental methods. This booklet is intended to be read alongside the companion QCA booklet *Teaching mental calculation strategies*.

Written work can take many different forms including:

- pictorial recording;
- informal jottings that help the learner but are not easily read by anyone else;
- words describing a mental calculation;
- use of appropriate signs and symbols;
- use of increasingly efficient standard methods.

Structure and organisation of this booklet

Part 1 lists the expectations set out in the *Framework for teaching mathematics* for paper and pencil recording in number. The focus of number work from reception to the end of year 3 should be on developing mental methods and on the recording of these methods in some form of written format. Subsequently, children will be taught more formal methods of written recording. The expectation is that by the end of year 6 children will be able to use a standard written method for each of the operations of addition, subtraction, multiplication and division.

Part 2 discusses the place of written methods in the mathematics curriculum and attempts to answer some questions about when and how they might be taught.

Part 3 is concerned with the development of written methods for addition and subtraction; it is in three sections:

- reception and key stage 1;
- years 3 and 4;
- years 5 and 6.

Part 4 has the same structure as Part 3 and relates to the development of written methods in multiplication and division.

Part 5 considers written methods that relate to the linking of fractions, decimals and percentages.

A statutory requirement of the national curriculum is that children should be taught to use and apply mathematics. Thus, where possible, number work should be set in context. Teachers might introduce problems involving money or measurement. Such problems can be discussed with all the children in the class at the start of the lesson and methods developed for them to do and record the appropriate calculations. Differentiation can then be provided by allowing each child to work with written methods that they have fully understood and mastered. In the final plenary session, children can share and compare their methods and key ideas can be drawn together.

Part 1

Written recording and calculation strategies: expectations for each year

This section sets out some of the important mental calculation and oral skills, knowledge and understanding which underpin written calculations. These are drawn mainly from the key objectives in the *Framework for teaching mathematics*, and are fully aligned with the expectations of the revised national curriculum.

Underpinning mental and oral skills, knowledge and understanding

Written recording and calculation strategies

Reception

- say and use the number names in order in familiar contexts
- count reliably up to 10 everyday objects
- recognise numerals 1 to 9
- use language such as more or less, greater or smaller, heavier or lighter to compare two numbers or quantities
- in practical activities and discussions, begin to use vocabulary involved in adding and subtracting
- find one more or one less than a number from 1 to 10
- begin to relate addition to combining two groups of objects, and subtraction to ‘taking away’
- use developing mathematical ideas and methods to solve practical problems

- begin to record in the context of play or practical activities and problems *eg recording using objects, marks, stamps, etc; writing shopping bills in the class shop; recording how many children in the class come to school on the bus*

Year 1

- count reliably at least 20 objects
- count on and back in ones from any small number, and in tens from and back to zero
- read, write and order numbers from 0 to at least 20; understand and use the vocabulary of comparing and ordering these numbers
- within the range of 0 to 30, say the number that is 1 or 10 more than any given number
- understand the operation of addition and subtraction (as ‘take away’ or ‘difference’), and use the related vocabulary
- know by heart all pairs of numbers with a total of 10
- use mental strategies to solve simple problems using counting, addition, subtraction, doubling and halving, explaining methods and reasoning orally

- record in the context of practical activities and when solving simple number problems *eg beginning to use conventional signs and symbols in number sentences to record mentally adding 11 and 12; recording the results of a survey of children’s favourite toys*

Underpinning mental and oral skills, knowledge and understanding

Year 2

- count, read and write numbers to at least 100; know what each digit represents (including 0 as a place holder)
- understand that subtraction is the inverse of addition; state the subtraction corresponding to a given addition and vice versa
- know by heart all the addition and subtraction facts for each number up to at least 10
- use knowledge that addition can be done in any order to do mental calculations more efficiently
- understand the operation of multiplication as repeated addition or as describing an array
- know by heart facts for the 2 and 10 times-tables
- choose and use appropriate operations and efficient calculation strategies to solve problems, explaining how the problem was solved

Written recording and calculation strategies

- develop recording in the context of practical work and in explaining how problems were solved
eg recording how much money could be in a box if there are five coins; recording the results of working on the 'handshake' problem
- use paper and pencil methods to support, record and explain mental addition and subtraction of numbers up to 100
eg recording what happens when 10 is added to any number up to 100; writing an explanation of how $93 - 89$ was calculated mentally; using correct signs and symbols to record number sentences such as $46 + 20 = 66$

Year 3

- read, write and order numbers to at least 1000; know what each digit represents
- partition a number into multiples of 10 and ones (TU) or multiples of 100, 10 and ones (HTU)
eg $537 = 500 + 30 + 7$
- count on and back in tens or hundreds from any two- or three-digit number
- know by heart all addition and subtraction facts for numbers up to 20
- add and subtract mentally a 'near multiple of 10' to or from a two-digit number
- know by heart facts for the 2, 5 and 10 times-tables
- understand division and recognise that division is the inverse of multiplication
- choose and use appropriate operations (including multiplication and division) to solve word problems, explaining methods and reasoning
- solve word problems involving numbers in 'real life' and explain how the problem was solved

- use informal paper and pencil methods to support, record and explain mental addition and subtraction of numbers up to 1000
eg using an empty number line to show how $301 - 45$ was calculated mentally
- begin to use column addition and subtraction, using expanded form
eg $456 + 63$:
$$\begin{array}{r} 400 + 50 + 6 \\ + \quad 60 + 3 \\ \hline 400 + 110 + 9 \\ = 519 \end{array}$$
- explain methods and reasoning, where appropriate, in writing
eg explaining how the missing number in a calculation such as $47 + \square = 55$ or $704 - \square = 698$ was found

Year 4

Underpinning mental and oral skills, knowledge and understanding

- use known number facts and place value to add or subtract mentally, including any pair of two-digit whole numbers
- add and subtract multiples of 10, 100 or 1000 to two- and three-digit numbers, including crossing the 100 and 1000 boundary
eg $34 + 50 = 84$
 $116 + 500 = 616$
 $1356 - 400 = 956$
- know by heart multiplication facts for the 2, 3, 4, 5, and 10 times-tables and derive the related division facts
- choose and use appropriate number operations and ways of calculating to solve problems
- solve mathematical problems or puzzles, recognise and explain patterns and relationships, generalise and predict
- use all four operations to solve word problems involving numbers in 'real life'

Written recording and calculation strategies

- develop and refine written methods for column addition and subtraction of two whole numbers less than 1000, and addition of more than two such numbers
- approximate first. Use informal pencil and paper methods to support, record or explain multiplications and divisions
- develop and refine written methods for $TU \times U$ and $TU \div U$
- choose and use appropriate ways of calculating (mental, mental with jottings, pencil and paper) to solve problems

Year 5

- multiply and divide any positive integer up to 10000 by 10 or 100 and understand the effect
- use decimal notation for tenths and hundredths
- relate fractions to division and to their decimal representations
- recognise equivalent additions
eg *adding 397 is equivalent to adding 400 and subtracting 3*
- calculate mentally a difference such as $8006 - 2993$
- use known number facts for mental addition and subtraction
eg $470 + 380$, $810 - 380$, $9.2 - 8.6$
- know by heart all multiplication facts up to 10×10 , and derive corresponding division facts; use known facts and place value to multiply and divide mentally
- use all four operations to solve word problems involving numbers and quantities in 'real life', including time, explaining methods and reasoning

- extend written methods to column addition and subtraction of two integers less than 1000
eg 454 376
 $+ 362$ $- 169$
- short multiplication of HTU by U
- long multiplication of TU by TU
- short division of HTU by U
- choose and use appropriate ways of calculating (mental, mental with jottings, written methods, calculator)
- explain methods and reasoning in writing

Underpinning mental and oral skills, knowledge and understanding

- multiply and divide decimals mentally by 10 or 100, and integers by 1000, and explain the effect
- use a fraction as an operator to find fractions of numbers or quantities
eg $\frac{5}{8}$ of 32, $\frac{9}{100}$ of 400 centimetres
- derive quickly division facts corresponding to tables up to 10×10
- solve simple problems involving ratio and proportion
- identify and use appropriate operations (including combinations of operations) to solve word problems involving numbers and quantities
- explain methods and reasoning

Written recording and calculation strategies

- extend written calculation methods to: column addition and subtraction of numbers involving decimals; short multiplication and division of numbers involving decimals; long multiplication of a three-digit by a two-digit integer
- in solving mathematical problems and problems involving 'real life', explain methods and reasoning in writing
- begin to develop from explaining a generalised relationship in words to expressing it in a formula using letters and symbols

Part 2

The role of written calculations

Introduction

Mental calculation, the ability to calculate ‘in your head’ is an important part of mathematics and an important part of coping with society’s demands and managing everyday events. *Teaching mental calculation strategies*, published by QCA, offers guidance to teachers on helping children develop skills of mental calculation.

As calculations become more complex, written methods become more important. Recording in mathematics, and in calculation in particular, is an important tool both for furthering the understanding of ideas and for communicating those ideas to others. There is, however, strong evidence, particularly from countries where standards are, at present, higher than those in the UK, that the introduction of written methods too early can undermine children’s fluency with number.

This booklet deals with when and how written methods should be introduced and how they link with mental strategies. This section addresses some of the questions often asked about written calculations.

Questions about written calculations

1) What do we mean by phrases such as ‘written methods’ and ‘written calculations’?

There are many phrases that are commonly used to describe the writing that accompanies a calculation, such as ‘pencil and paper procedures’, ‘written calculations’, ‘written methods’, ‘formal written methods’, ‘standard written algorithms’ and ‘personal written jottings’. In this booklet, a written method can be thought of as a structured annotation of a calculation, distinct from an informal jotting which nobody needs to see. A jotting can eventually be discarded, whereas a genuine written method has lasting value.

A useful written method is one that helps children to carry out a calculation and can be understood by others.

2) Why do we need written methods?

Written methods are complementary to mental methods and should not be seen as separate from them. As a long-term aim, children should be able to choose an efficient method – mental, written or calculator – that is appropriate to a given task.

There are a number of reasons why written methods may be useful. They can:

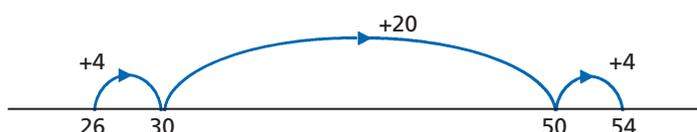
- represent work that has been done practically;
- support mental calculations – often in the form of jottings;
- record and explain mental calculations;

- help in observing patterns;
- communicate ideas and information;
- establish connections between practical experiences, symbols, language and patterns;
- help keep track of steps in longer tasks;
- develop mental imagery;
- work out calculations which are too difficult to do wholly mentally;
- develop efficiency in calculation;
- provide a means of practising a new concept or skill;
- help prepare children for calculations in algebra that they will meet in key stage 3.

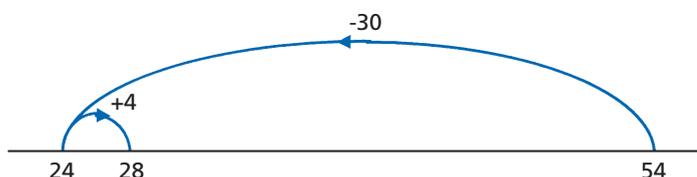
Written recording is needed to help us keep track of where we are in our calculation and to help explain our thinking or method to someone else.

3) Which is more important – written or mental calculation?

In every written method there is an element of mental processing. The revised national curriculum and the *Framework for teaching mathematics* emphasise the importance of developing strategies for mental calculation, with written calculation being reserved for those calculations that cannot be carried out mentally. Children may need to record jottings on the way to being able to carry out a mental calculation entirely in their head. Such recording might include the use of an empty number line. For example, to calculate $54 - 26$ as a difference, children might be shown how to use a mental strategy of bridging to useful ‘landmarks’, such as 30 and 50 and might record this in the written form:



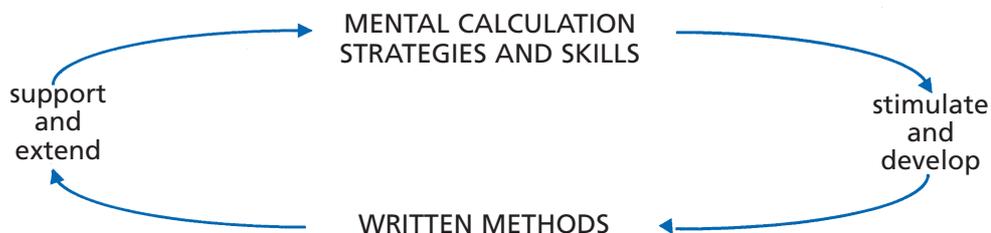
To calculate $54 - 26$ as a take-away, they might record:



In these examples, paper and pencil are used to record a mental strategy by showing appropriate ‘jumps’ on an empty number line. Later, children might be encouraged to imagine the number line and use the same mental methods. These can be recorded using equations written correctly with numerals, signs and symbols, eg

$$\begin{array}{ll}
 26 + 4 = 30 & \text{or} & 54 - 30 = 24 \\
 30 + 24 = 54 & & 54 - 26 = 54 - 30 + 4 \\
 54 - 26 = 28 & & 54 - 26 = 24 + 4 = 28
 \end{array}$$

Sharing written methods with the teacher encourages children to think about the mental strategies that underpin them and to develop new ideas. Written recording helps children to clarify their thinking. As children’s mental methods become more sophisticated, they will be able to use written methods that are more concise and more structured, and at this point they should begin to discard their informal methods and jottings. The development of written methods will support and extend the development of more fluent and sophisticated mental strategies. This is illustrated by the cycle:



Many published schemes up to now have been largely concerned with written calculation. When children are working with such schemes it might be necessary to leave out a number of the examples, or to encourage children to see many of them as potential mental calculations.

4) When do children need to start recording?

Children should be encouraged to see mathematics as a written as well as a spoken language. Teachers need to support and guide children through the following important stages:

- developing the use of pictures and a mixture of words and symbols to represent numerical activities;
- use of standard symbols and conventions, such as numerals 0 to 9, the equals sign and the operations signs to record mental calculations;
- use of jottings to aid a mental strategy;
- use of expanded forms of recording as a step towards standard paper and pencil methods;
- use of compact forms of recording;
- use of a calculator.

Jottings should be discarded when a child has a secure understanding of a mental method. Children will go through these stages, albeit at varying rates, throughout key stages 1 and 2.

Reception	Year 1	Year 2	Year 3	Year 4	Year 5	Year 6
MAKING A RECORD OF A CALCULATION						
JOTTING TO SUPPORT A NEW MENTAL STRATEGY (BUT DITCHING JOTTING WHEN SECURE IN THAT STRATEGY)						
EXPLAINING A MENTAL STRATEGY						
DEVELOPING WRITTEN METHODS						

It is important to encourage children to look first at the problem and then get them to decide which is the best method to choose – pictures, mental calculation with or without jottings, structured recording or calculator.

5) When are written methods best introduced?

At first, children's recordings may not be easy for someone else to interpret, but they form an important stage in developing fluency. During year 1, children should begin to use the +, - and = signs to record their mental calculations in a number sentence. When children start to work with larger numbers and to carry out calculations with several steps, it will be harder for them to hold all the information in their head, but they need to be thoroughly secure in a range of mental strategies before they are ready to begin to use more formal written methods. Children attempting to use formal written methods without a secure understanding will try to remember rules, which may result in unnecessary and mistaken applications of a standard method, such as in the following two examples:

James wanted to calculate 16 - 9. He wrote:

$$\begin{array}{r} 0 \\ \times 16 \\ - 9 \\ \hline 7 \end{array}$$

He explained his working as 'Six take away nine, you can't, so borrow one from the tens, so now it's sixteen take away nine and that's seven.'

In the end, James carried out the calculation in his head. If a written recording of such a calculation is desired, then the horizontal written form $16 - 9 = 7$ is appropriate.

Emma, wanted to add 24 and 39. She wrote:

$$\begin{array}{r} 24 \\ + 39 \\ \hline 513 \end{array}$$

Emma has clearly not considered the value of the whole numbers involved. She apparently tried to remember a rule she had been taught for writing down an addition, rather than thinking about the addition process itself. Children at level 3 should be able to add or subtract any pair of two-digit numbers mentally. Recording such calculations vertically may not be helpful in achieving this. Using mental strategies Emma might have partitioned the numbers into tens and ones, and recorded:

$$24 + 39 = 20 + 30 + 4 + 9 = 50 + 13 = 63.$$

Alternatively, she might have noticed that 39 is close to 40 and written:

$$24 + 39 = 24 + 40 - 1 = 64 - 1 = 63$$

Emma might not have made her error if she had been trained to say the numbers to herself rather than concentrate her attention on the individual digits in those numbers, so losing sight of what to reasonably expect as the answer.

6) Should children be taught one standard method for each operation?

The aim should be that by the end of year 6 children will have been taught, and be secure with, a compact standard method for each operation. There are a number of written methods which work, some of which are generally accepted as 'standard' in this country. Each school should choose one method to develop for each operation. If schools in the same area arrange to teach the same method this will facilitate their children's transition to secondary school mathematics. Children develop understanding at different rates so it is to be expected that, at any time, in any class, there will be children working at expanded forms of the same method, at a range of levels of efficiency and understanding. They need to be guided appropriately towards increased efficiency in their writing and to less dependence on informal jottings. It is important to talk to and listen to children so that written methods are developed in a way that helps them to develop efficiency while maintaining understanding.

Children should work towards knowing and understanding a compact standard method for each numerical operation so that they are secure with these by the end of year 6.

There are advantages in using standard written methods for calculations:

- they will always give the correct answer if applied correctly;
- they are efficient;
- they can be applied to any numbers, although they are not necessarily the best method for all numbers even if large;
- eventually, when a standard method is fully understood, it is possible to carry it out automatically, so that concentration can be directed towards the development of new ideas in which that particular calculation plays a minor role.

However, their condensed standard form can be difficult to understand. Because the methods are condensed, they do not easily display the underlying mathematical rules that are being used. Children are prone to error when they use mechanically methods they do not fully understand.

7) When should we start vertical calculations?

At key stage 1, calculations should be recorded in horizontal form, so that the written record closely resembles the way in which the children calculate mentally and would describe their working. The *Framework for teaching mathematics* recommends that horizontal recording should be emphasised until well into year 3 and then extended methods that with a vertical layout which models the calculation process should be introduced. An important goal is for children to be able to record clearly how they carry out a calculation. When children start to record multiplication and division, they may need to use a variety of methods that help them to understand the nature of the operations. Vertical recording should be delayed until children are secure in their knowledge of basic facts and the way they may be used to derive new ones.

8) Which calculations should be done using a vertical format?

This will depend on the numbers involved and the individual child.

When faced with a calculation, no matter how large or difficult the numbers may appear to be, all children should ask themselves:

- ‘Can I do this in my head?’
- ‘Do I know the approximate size of the answer?’
- ‘If I can’t do it wholly in my head, what do I need to write down in order to help me calculate the answer?’

and only then

- ‘Will the written method I know be helpful?’

If children use standard written methods uncritically, they may not notice that a particular calculation could be carried out mentally or in a much simpler way, as shown by Alice’s unsuccessful attempt to use a vertical format to subtract £1.99 from £2.00.

$$\begin{array}{r} \text{£}2.00 \\ - \text{£}1.99 \\ \hline \text{£}0.09 \end{array}$$

A vertical layout is sometimes unhelpful when zeros are involved. Examples of calculations which should be done mentally even when vertical formats have been introduced are $301 + 141$, $1200 + 3600$, $3001 - 2997$, 19×20 and $540 + 6$, as well as multiplication and division by powers of 10 and multiplication by 25 or 50. Some children will also find a mental method useful for other calculations, for example:

$$24 \times 16 = (25 \times 16) - (1 \times 16)$$

$$25 \times 16 = 50 \times 8 = 100 \times 4 \text{ (using doubling and halving twice)}$$

$$24 \times 16 = 384$$

or by doubling

$$24 \times 2 = 48$$

$$24 \times 4 = 96$$

$$24 \times 8 = 192$$

$$24 \times 16 = 384$$

Find the change from £5 if you spend £2.37.

Again, the zeros in this calculation may be the chief source of confusion if the traditional vertical layout is used. The obvious mental strategy is to build £2.37 up to £2.40, then up to £2.50, then to £3.00 and then to £5.00, the actual amount of change found by adding 3p + 10p + 50p + £2.00. This could be recorded as:

$$\begin{aligned}£2.37 + 3p &= £2.40 \\£2.40 + 60p &= £3 \\£3 + £2 &= £5 \\3p + 60p + £2 &= £2.63\end{aligned}$$

An example of a sensible use of vertical format is the addition

$$\begin{array}{r}28.07 \\+ 13.95 \\ \hline 42.02\end{array}$$

9) How important is it to practice written methods?

With a greater emphasis on developing secure mental methods, less time should be needed for teaching written methods. However, practice is necessary to improve proficiency and to increase confidence in the learning of any new skill. Children should not automatically be expected to complete all questions in every exercise in a textbook. Those who are successful at a particular written method would be better occupied doing more challenging work; those that get a string of questions wrong will gain nothing positive from the experience and may even have their misconceptions reinforced. Practice of a new skill should be set in a variety of contexts, little and often, over a period of time. Appropriate work should be set for children who are at different developmental stages so they are working at a level that reinforces their skill and understanding.

Practice should consolidate both skills and understanding. Practice is not just a case of repetition of many similar examples.

Practice can be achieved in a variety of ways:

- playing a number game;
- making up calculations for other people;
- working through a few (six to eight) set examples;
- teaching someone else a strategy recently encountered;
- choosing different numbers for oneself to repeat a particular type of calculation;
- looking at, and commenting on, each other's work.

Teachers should distinguish between practice for the whole class (in which children can share ideas) and practice for individual consolidation (which might be homework).

10) What about children at different stages of attainment?

In many classes, children will be at different stages in their move towards efficiency. This variety can be a good basis for discussion, and the sharing of ideas can itself help those who are using an inefficient method to recognise and to work towards using more efficient strategies. Discussions with children will provide opportunities to make a diagnostic assessment of their understanding and, if appropriate, move them to a more compact form of recording. This process should not be rushed; children should be moved on when they are ready. The objective is that all children should reach the stage of compact recording by year 6.

Children can be encouraged to select numbers for their own calculations. This choice may need to be guided so they recognise that the method they use may be determined by the numbers involved and that in some cases even large numbers are more easily dealt with mentally with a different form of written recording.

11) How can children's readiness for written calculations be judged?

A teacher will have to make fine judgements as to when a class is ready to move towards the formal recording of calculations. As this booklet makes clear, some form of recording will always be taking place in the classroom. However, the development of expanded written formats with progression to an efficient standard written method relies on children having mastered a range of mental calculation skills.

Judgements will need to be made as to whether children in a class possess sufficient of these skills to progress, and different criteria should be used to assess readiness for learning written methods for addition and subtraction as opposed to readiness to learn written methods for multiplication and division.

A short list of criteria for readiness for formal written methods of addition and subtraction would include:

- Do the children know addition and subtraction facts to 20?
- Do they understand place value and can they partition numbers into hundreds, tens and ones?
- Do they use and apply the commutative and associative laws of addition?
- Can they add at least three single-digit numbers mentally?
- Can they add and subtract any pair of two-digit numbers mentally?
- Can they explain their mental strategies orally and record them using informal jottings?

Corresponding criteria to indicate readiness to learn formal written methods for multiplication and division are:

- Do the children know the 2, 3, 4, 5 and 10 times-tables and the corresponding division facts?
- Do they know the result of multiplying by 0 or 1?
 - do they understand place value?
 - do they understand 0 as a place holder?
 - can they multiply two- and three-digit numbers mentally by 10 and 100?

- Can they use their knowledge of all the multiplication tables to approximate products and quotients using powers of 10?
- Do they use the commutative and associative laws for multiplication, and the distributive law of multiplication over addition or subtraction?
- Can they double and halve two-digit numbers mentally?
- Can they use multiplication facts they know to derive mentally other multiplication facts that they do not know.
- Can they explain their mental strategies orally and record them using informal jottings?

These checklists are not intended to be exhaustive. They are merely a guide for the teacher to decide when a class is ready to start to move from informal methods of recording calculations to more refined formal written methods, with the ultimate objective of teaching a standard written method of each operation by the end of year 6.

In deciding whether children are ready to move on, teachers need to be sure that they are not making judgements that will adversely affect the children's progress. There may be children in the class who are not making the progress expected of them, but who might make progress by being moved on. Some children may be denied this entitlement if a teacher follows criteria in a slavish manner and holds them back; the danger exists that the children will always be held back and may never be given the opportunity to progress.

The teacher must make formative assessment judgements about children's grasp of mental calculation strategies before deciding whether to begin teaching formal written calculation.

12) What resources are needed?

There are no specific resources, apart from paper and pencil, needed to teach written calculations, but multibase blocks, number tracks and lines and an abacus might be useful aids to understanding for some children. Schools may be using textbooks that under-emphasise the importance of mental calculation and over-emphasise the use of written recording. These will need to be adapted to match the progress of children in line with the expectations of the national curriculum.

13) Is the language we use important?

It may not always be appropriate to correct children when they misuse mathematical language. However, teachers should make sure that when they themselves are talking about mathematics they use correct terminology and do not introduce words which may be misleading or unhelpful. When manipulating numbers greater than nine, it is important to refer to the actual value of the number the digits represent. If using informal language such as 'carry' or 'exchange' as shorthand, it is important that children are able to explain in their own words the mathematical processes to which the words relate. The *Framework for teaching mathematics* and the booklet *Mathematical vocabulary* provide guidance on the range and variety of language that teachers and children should use in order to broaden the mathematical vocabulary used in mathematics lessons.

14) What about parents' expectations?

For many parents, the word 'mathematics' conjures up their own experience of many pages of written 'sums' and perhaps of being punished for not knowing the multiplication tables. They may feel anxious about the change of emphasis towards mental strategies and the consequent fact that written methods will occur later than they have come to expect. Many parents may even think that they learned formal written methods before they actually did! What parents need to understand is that by learning mental strategies children will be able to tackle complex and challenging problems without the constraints of formal written methods. Delay in introducing written recording will not result in a drop in standards. It will be very important for schools to share the philosophy behind these changes with parents and to involve them, as far as is possible, with their child's mathematical development. These opportunities could be in the form of parents' evenings, involving parents in mathematical assemblies, mathematics days, homework clubs or through the loan of mathematical games to play at home. Parents should understand that children will need to start with unstructured personal jottings that should be valued. These jottings are an essential step towards conceptual understanding. As children gain in understanding, their written methods will become more fluent and more efficient.

Many children successfully learn column addition and subtraction by year 4. However, many children need more time to understand the complexity of these operations. These children need experience of expanded written methods before they can progress to more compact forms. Parents can help their children to consolidate their understanding of expanded written methods so that they can perform column addition and subtraction by year 6.

Part 3

Addition and subtraction

Introduction

This chapter shows how children's written recording of addition and subtraction develops throughout key stages 1 and 2. This recording will take the form of:

- jottings, to support a mental calculation;
- a record of a mental strategy, written in a way that helps someone else to understand;
- extended methods of calculation that prepare the way for increasingly efficient written calculations that cannot easily be done mentally.

It is important to introduce a gradient of difficulty for examples using particular methods of addition or subtraction. At each stage of development children should reach their full potential. The teacher's task is to help them utilise their knowledge, skills and understanding to maximum effect. Children should be presented with problems ranging in difficulty, that allow them to practise using a common structured method for arriving at the answer. Children also need experience of problem solving in which they have to choose the appropriate method of solution.

Reception and key stage 1

The *Framework for teaching mathematics* makes it clear that children are not expected to use paper and pencil procedures for addition or subtraction in key stage 1. At this stage, their experience of these operations will be a mixture of practical, oral and mental work. They will, however, be making use of written forms to:

- make a record in pictures, words or symbols of addition or subtraction activities that they have already carried out, and to construct number sentences;
- explain to someone else what they have done;
- interpret information that requires practical, oral or mental calculation;
- begin to read records made by their teacher;
- help work out steps in a calculation they will later do entirely mentally.

Much of young children's work with addition and subtraction will be oral, arising out of practical activities. Talking about what they have done is an essential precursor to written recording. Children will need to have plenty of experience of using their own individual ways of recording addition and subtraction activities before they begin to record more formally.

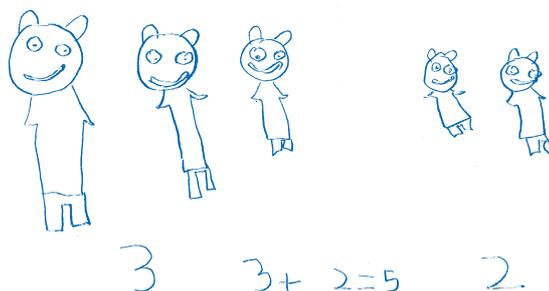
Children should be encouraged to talk and write about their work in their own way. When their understanding is sound, conventional labels and symbols can easily be introduced; trying to do it the other way round – to start with the symbol and then try to explain it – is much harder. It is easy to be misled, by children who start to use standard forms of recording too early, into thinking that they necessarily understand what they have written.

Children's individual recording

Children's written recording of addition and subtraction will probably begin with simple pictorial representation of what they have done practically. These are easily understood if teachers know what the children have actually been doing, but they will not, generally, be a form of recording that anyone else could understand. Sharing a range of different ways of recording will help children choose the most appropriate method.

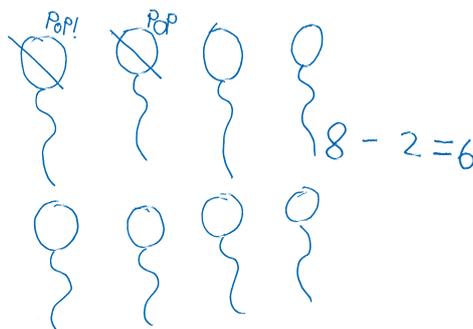
For addition, children will need to be taught to record the outcome when two groups of objects are combined into one group:

Jane had 3 bears. She was given 2 more. How many does she have now?



In the case of subtraction, children need practical activities of 'taking away', that is finding how many are left from a collection of objects when some are removed, and also of 'finding the difference' which involves making a comparison between the numbers in two groups of objects. They need to recognise that both are examples of subtraction and that the same symbol can be used to describe them.

There were 8 balloons. Two popped. How many are left?



How many more biscuits does Sally have than you?

The biscuits were represented by counters:



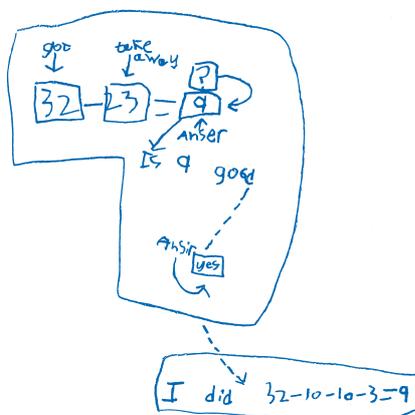
which led to the response: 'Sally has 3 more biscuits than me.'

Her teacher wrote $5 + 3 = 8$ to emphasise that five and three more is eight.

A mixture of words and symbols will be used by children in order to explain to someone else the mental methods they have used. Children will use a variety of ways of recording addition and subtraction reflecting the variety of mental methods used.

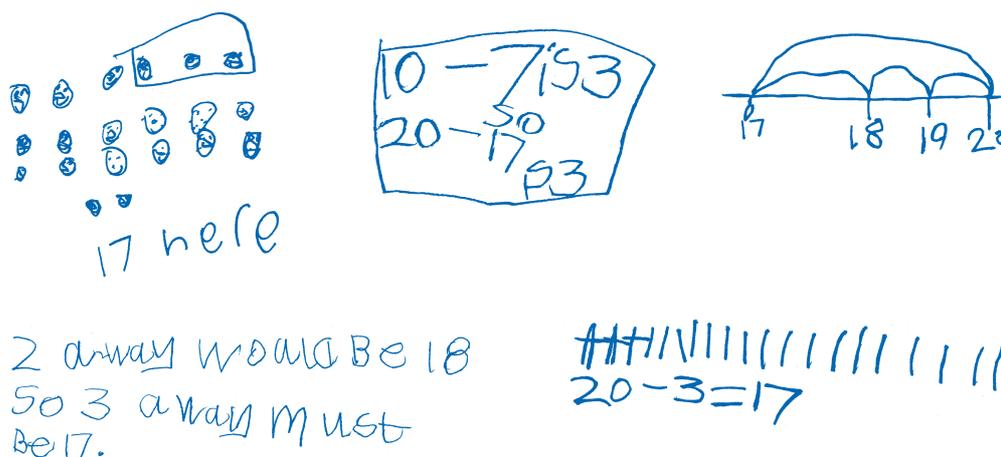
There are 32 sweets in the tube, and 23 children in the class. Everybody is given one sweet. How many are left over?

This child has not only found the answer, but also checked that he thought it was a 'good' answer:



Different methods are often used for the same calculation and this can lead to useful discussion:

There are 20 children in our class. Three are away today. How many are here?



Teachers can take the opportunity, when appropriate, to 'scribe' children's oral accounts and to introduce the use of appropriate symbols.

For example, Francesca, who was adding 12 and 13, said:

'I know that twelve and thirteen is twenty-five because twelve is ten and two and thirteen is ten and three, so it's ten and ten and two and three which is twenty and five, twenty-five!'

Her teacher wrote:

$$\begin{aligned}
 12 + 13 &= 10 + 2 + 10 + 3 \\
 &= 10 + 10 + 2 + 3 \\
 &= 20 + 5 \\
 &= 25
 \end{aligned}$$

and discussed this form of recording with her.

With other children she might have scribed the same calculation as:



It is important that children see a range of ways to record mental methods.

Children's understanding of the symbols for addition and equals can be checked by asking questions such as:

What could $5 + 2 = 7$ mean?

Five people went out to look for two friends. They brought them back to the house to make seven.

Using a number track and number line

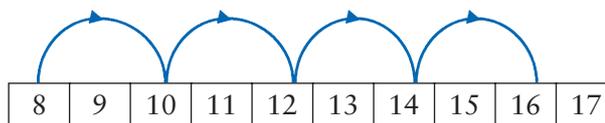
The use of number tracks and lines is very helpful for teaching children the order of numbers and for images of addition and subtraction. In reception classes, it may begin with children physically jumping forwards and backwards along a track which has the numbers marked on each space.

Find a way to record on this number track what happened when you were standing on 5 and you jumped back 3 spaces.



Children can then use the track for finding patterns:

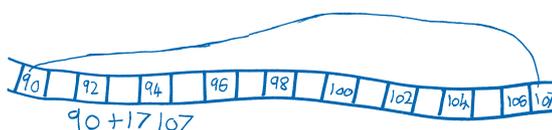
Mark the numbers you land on when you hop in twos from different starting numbers. What happens if you jump back in twos instead?



'I jumped in twos from 8, that's 8, 10, 12, 14, 16.'

and for larger numbers, but still counting on:

Choose two numbers to add on the number track:



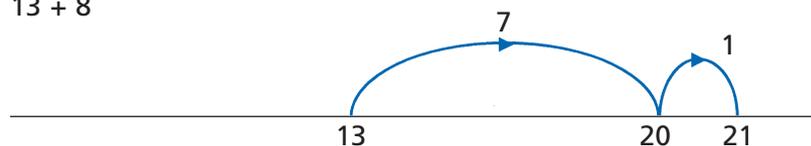
The transition from the use of number tracks to number lines is important. With number tracks, spaces are marked; the child will typically place a counter on a square with a numeral on it and count forwards or backwards a number of spaces, as in the previous example of hopping in twos. On number lines points are marked, allowing fractions to be placed between the whole numbers – for example $6\frac{1}{2}$ can be placed halfway between 6 and 7 – leading to the later development of the idea of continuity of number. Both number tracks and number lines can support the idea of the numbers going on for ever in both directions.

Empty lines have no numbers marked and children write in the numbers they have used. Children can use empty number lines to illustrate their mental methods. The facility with which a child uses number lines can be used by the teacher to assess the child's progress.

Empty number lines provide a means of supporting and developing children's counting strategies with a pictorial recording.

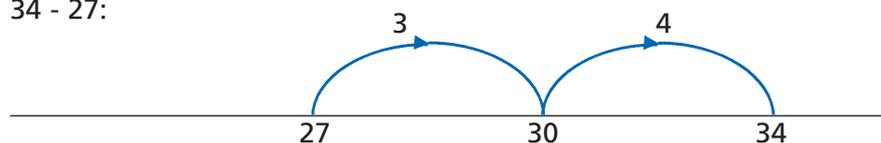
There are 13 boys and 8 girls in the room. How many altogether?

$$13 + 8$$



There are 34 children in the classroom. 27 go to the hall. How many are left?

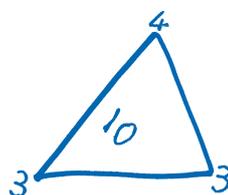
$$34 - 27:$$

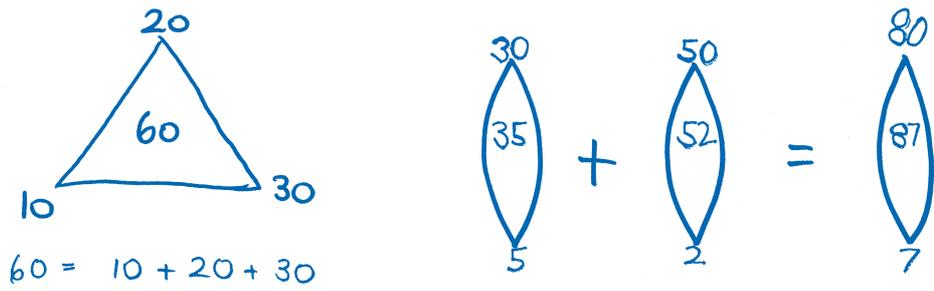


Introducing new recording formats

By the end of year 2, children should know whether addition or subtraction is to be used when solving a problem given in words and be able to choose suitable mental strategies to find a solution. They should be using symbols as a means of recording mental calculations and as a means of explaining their thinking to someone else. Teachers can suggest possible ways to record working.

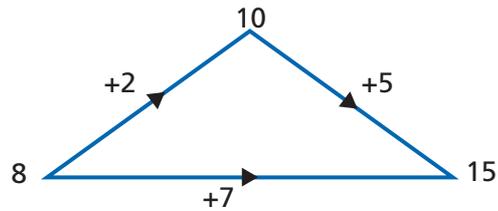
For example, partitioning can be introduced by inviting young children to make a tower of seven bricks and then to split the tower into three groups in any way they like. With larger numbers, children should be encouraged to show the tens and units separately. Other children can use the same type of display to choose their own numbers and the way in which they will split them up:





Other recording styles that teachers can introduce include:

8 + 7



This encourages building up to ten.

Some children will be able to work mentally with three-digit numbers partitioning into hundreds, tens and units.

Use of empty boxes

Empty boxes provide a useful recording structure for exploring the inverse operation relationship between addition and subtraction. Their use can be a basis for some early algebraic ideas and help children to develop a better understanding of the equals sign. Children can start with examples such as:

3 + 4 = □, moving on to

13 + □ = 17, which can be turned into a subtraction: □ = 17 - 13

and move to examples that explore the tens and units structure of numbers:

$$\boxed{20} + 15 = \boxed{35}$$

or by giving one missing number:

12 = 10 + □ 15 = 10 + □ 28 = 20 + □

Examples such as □ + □ = □ - □ require children to understand that the equals sign indicates numerical balance. It is better to use descriptions such as ‘the same as’ or ‘balances’ to describe the equals sign, rather than ‘makes’. ‘Makes’ only has meaning if the calculation to be done is on the left of the equals sign and the ‘answer’ on the right. It is not appropriate for statements such as:

$$\boxed{20} + \boxed{2} = \boxed{24} - \boxed{2}$$

Use of symbols

The '=' symbol should be read as 'equals'. In this example, it is important to stress that in the previous example $20 + 2$ and $24 - 2$ both have the same value 22.

Children should also be introduced to the inequality symbols for greater than '>' and less than '<', recognising that the equals sign 'closes up' if the number on that side is less than the number on the other side and 'opens' if it is greater. They can be asked to insert '=', '>' or '<' between pairs of numbers or expressions.

$$\begin{array}{rcl} 25 & < & 32 \\ 17 & & 15 \\ 48 & & 20 + 27 \\ 34 - 8 & & 26 \\ 6 + 7 & & 19 - 6 \end{array}$$

Adapting published resources

Many existing published resources can be adapted and used more flexibly. For example, teachers can:

- use only those pages that relate to the teaching objectives, drawing from the materials from any year;
- encourage children to decide which questions they will do mentally;
- invite children to make up additional examples;
- look for different questions that use the same number facts.

Years 3 and 4

In years 3 and 4, children will continue to use written recording of addition and subtraction calculations to support their mental calculations. They should use mathematically correct equations to record their mental calculations in a way which explains the steps they have taken. Alongside these, they should begin to develop written methods, involving column addition and subtraction for those calculations they cannot do in their head.

Addition and subtraction of two 2-digit numbers

Most children will learn to add and subtract two 2-digit numbers entirely mentally, choosing the best strategy for the numbers involved. Column addition or subtraction methods are not usually appropriate for pairs of two-digit numbers, except when written methods are first introduced to demonstrate what is already known so as to move on quickly to work with three-digit numbers. Mental calculations can be recorded in a variety of ways.

Written records to support mental methods have three purposes:

- they reinforce the mental strategy of the child who is writing;
- they provide teachers with evidence to enable them to assess children's progress;
- they enable children to share their ideas with each other.

Take turns to turn over four single-digit cards to make two 2-digit numbers and then add them together. Write down your calculations so that other children understand what you have done.

'I jumped in tens on the number line in my head.'

$$\begin{array}{l} 24 + 27 = \\ 25 + 25 = 50 \quad 50 + 1 = 51 \end{array} \qquad 63 + 30 = 60 + 30 + 3 = 93$$

$$24 + 27 = 25 + 25 + 1 = 50 + 1 = 51$$

Their recording will show how children respond to the same task using different mental strategies. Different mental strategies will also be used depending on the numbers involved.

$$\begin{aligned} 29 + 39 &= 30 + 40 - 2 \\ &= 70 - 2 \\ &= 68 \end{aligned}$$

$$37 + 26 = 40 + 23 = 63$$

$$39 + 45 = 30 + 8 + 40 + 5 = 70 + 13 = 83$$

If children have been encouraged to use a range of mental methods, they should not find any difficulty in extending them to numbers crossing the hundreds boundary.

In a school there are 76 boys and 93 girls. How many children are there altogether?

$$\begin{array}{l} 93 + 76 = 90 + 70 + 3 + 6 \\ = 90 + 70 + 9 \\ = 160 + 9 \\ = 169 \end{array} \qquad \begin{array}{l} 93 + 76 = 93 + 7 + 76 - 7 \\ = 100 + 69 \\ = 169 \end{array}$$

There are 76 boys and 93 girls in the school. Find how many more girls there are than boys?

One strategy might be recorded as:

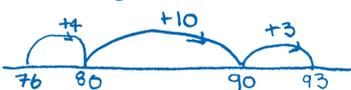
$$\begin{aligned} &93 - 76 \\ &93 - 73 = 20 \\ \text{So } &93 - 76 = 17 \end{aligned}$$

Although this is a comparison or difference form of subtraction, many will use a 'counting on' method, turning the question into $76 + \square = 93$, either with or without using a number line:

$$76 + \square = 93$$

$$76 + 4 = 80$$

$$80 + 10 = 90$$

$$90 + 3 = 93$$


$$76 + \boxed{17} = 93$$

How many children in a school of 90 children have gone on a school trip if 37 children are left in school and none is absent?

This problem can be seen as $90 - \square = 37$. An important principle of subtraction to discuss with children is that this is equivalent to, and can be recorded as, $90 - 37 = \square$

$$90 - 37$$

$$90 - 40 = 50$$

so $90 - 37 = 53$

Vertical layouts

Most children in year 4 will be able to extend mental methods such as those illustrated above to three-digit numbers. But it is around this level of complexity that sometimes, with some questions, children will begin to find a vertical layout a better way of tackling the calculation. As more digits are involved, a system that lines up the hundreds, tens and ones becomes helpful in keeping track of what is being added to what or subtracted from what. **This should be introduced as an approach for those calculations that are too complicated to deal with mentally.**

There are still many cases with large numbers when a mental strategy is more appropriate: it would be a backward step for children to use a vertical layout to calculate, for example, $500 + 123$ or $746 - 300$ or $401 - 397$.

Children must learn to look at the numbers involved carefully to decide if a mental strategy is appropriate even when the calculation is presented to them in a vertical format.

Addition

All teachers are familiar with the traditional standard addition format, although there might be differences of opinion as to the positioning of the addition sign and the carry-digits. For example, it might be:

$$\begin{array}{r} 367 \\ + 256 \\ \hline 623 \\ \hline \end{array}$$

11

This is a condensed form of written recording and children using it can be prone to errors, usually because it encourages them to look only at the digits rather than to think of the numbers as a whole. The procedure of working from right to left conflicts with the mental strategies of starting with the biggest value digits and working from left to right. Children should be introduced first to expanded forms of vertical layout which build more naturally on mental strategies.

The examples below show a variety of ways in which children can be helped to develop addition skills, using what they already know to calculate ‘partial sums’ while introducing the idea of lining up the hundreds, tens and ones.

The blue team’s score of 287 points is to be increased by a further 45 points. What is the new score?

287 + 45 can be written as the partial sums:

$$\begin{array}{r}
 200 + 80 + 7 \\
 + \quad \quad 40 + 5 \\
 \hline
 200 + 120 + 12 = 332
 \end{array}$$

or

287 + 45

$$\begin{array}{r}
 200 \\
 80 + 40 = 120 \\
 7 + 5 = 12 \\
 \hline
 332
 \end{array}$$

Adding from left to right leads to:

$$\begin{array}{r}
 287 \quad \text{or} \quad 287 \\
 + \quad 45 \quad \quad + \quad 45 \\
 \hline
 200 \quad \quad \quad 12 \\
 120 \quad \quad \quad 120 \\
 \hline
 12 \quad \quad \quad 200 \\
 \hline
 332 \quad \quad \quad 332
 \end{array}$$

recording from left to right or from right to left and adding mentally the partial sums from the top or the bottom. The equivalence of these different approaches is a direct consequence of the commutativity of addition.

Children need to be competent at mentally adding strings of single-digit numbers, presented vertically, before attempting addition of larger numbers using a vertical format.

Adding strings of numbers

In years 3 and 4, children will move on from the addition of a string of single-digit numbers to the addition of a string of numbers with more than one digit. This will involve the repeated use of the methods developed above for the addition of two numbers, together with the same kind of expectations for recording. A useful additional technique to develop for the addition of a string of numbers is to pair

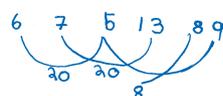
numbers together helpfully, such as those that combine to 10 or 11 or 20. Children can then be encouraged to go on grouping the reduced string of numbers in ways that add easily, using partitioning when necessary.

Calculate the total score in a game in which 6, 7, 5, 13, 8 and 9 points have been scored.

A year 3 child might record their successive simplifications of this addition as follows:



$$\begin{aligned}
 &= 11 + 9 + 20 + 8 \\
 &= 20 + 20 + 8 \\
 &= 48
 \end{aligned}$$



$$\begin{aligned}
 &= 11 + 9 + 20 + 8 \\
 &= 20 + 20 + 8 \\
 &= 48
 \end{aligned}$$

A child chooses items costing 34p, 12p, 17p, 28p and 46p from a catalogue. Work out whether these can be purchased with £1.50.

A year 4 child might first partition these prices into tens and ones ($34 = 30 + 4$, etc), and then mentally start adding the tens and the ones separately (eg $30 + 10 + 10 = 50$, $4 + 7 + 2 = 13$ so $34 + 12 + 17 = 63$). Alternatively the numbers can be added in pairs. In either case a written record is useful to keep track of the calculation:

$$34\text{p} + 46\text{p} = 80\text{p}$$

$$17\text{p} + 12\text{p} = 29\text{p}$$

$$109\text{p} + 28\text{p} = 137\text{p}$$

The items can be purchased with £1.50.

When understanding is secure, children should proceed to add sums of money in columns, knowing the decimal points should line up under each other.

Subtraction

As with addition, there are several possible vertical layouts for subtraction that can help children to line up hundreds, tens and ones. The standard decomposition layout for subtraction is well known. Also familiar are the many ways children make errors with what is a complex procedure.

As with the standard format for addition, if introduced too quickly this layout can lead to errors because of the tendency to see only digits rather than the numbers as a whole. The process is based on exchange. It is important to develop this idea through plenty of preliminary work on partitioning into hundreds, tens and ones. Base ten

materials or money can be used to illustrate exchanging a hundred for ten tens and a ten for ten ones.

A computer for the school costs £567. A jumble sale has raised £243. How much more has to be found?

$$567 - 243.$$

Most year 4 children would be able to carry this out in their head by a counting on method (complementary addition), for example, $243 + 7 = 250$, $250 + 50 = 300$, $300 + 200 = 500$, $500 + 67 = 567$; or perhaps by partitioning: $(500 + 60 + 7) - (200 + 40 + 3)$.

A relatively simple example like this where no exchange is necessary could be used to introduce children to the idea of lining up the hundreds, tens and ones:

$$567 - 243$$

$$\begin{array}{r} 500 + 60 + 7 \\ 200 + 40 + 3 \\ \hline 300 + 20 + 4 = 324 \end{array}$$

The price of a computer is reduced from £767 to £619. By how much is it reduced?

Working from left to right a child might record:

$$\begin{array}{r} 700 + 60 + 7 \\ - 600 + 10 + 9 \\ \hline 100 + 50 + -2 = 148 \end{array}$$

The recording of -2 does not imply a sophisticated understanding of negative numbers. It simply represents 'I can only take away 7, so there's still 2 to be taken away.' This child can be shown that working from right to left and exchanging is simpler and leads more directly to the answer.

In a similar manner, the calculation $567 - 378$ could be performed as follows:

$$\begin{array}{r} 500 + 60 + 7 \\ 300 + 70 + 8 \\ \hline 200 - 10 - 1 = 200 - 11 = 189 \end{array}$$

Children can then be introduced to situations where exchange is necessary and shown the importance of working from right to left.

Exploring examples with a layout using partitioning into hundreds, tens and units as follows should be used as an introductory stage to more condensed formats. For the example just given, the following steps can be identified:

First, looking at the units

$$\begin{array}{r} 500 + 60 + 7 \\ 300 + 70 + 8 \\ \hline ? \end{array}$$

Exchanging a ten for ten units, dealing with the units and then looking at the tens:

$$\begin{array}{r} 500 + \overset{50}{\cancel{60}} + 17 \\ \underline{300 + 70 + 8} \\ ? + 9 \end{array}$$

Then, exchanging a hundred for ten tens and completing the calculation:

$$\begin{array}{r} \overset{400}{\cancel{500}} + \overset{150}{\cancel{60}} + 17 \\ \underline{300 + 70 + 8} \\ 100 + 80 + 9 = 189 \end{array}$$

The calculation can be compacted into the form

$$\begin{array}{r} \overset{4}{\cancel{5}} \overset{15}{\cancel{6}} \overset{17}{\cancel{7}} \\ \underline{3 \ 7 \ 8} \\ 1 \ 8 \ 9 \end{array}$$

Zeros in the first number in a subtraction

Zeros in the first number in a subtraction can cause problems. Sometimes a calculation is best done mentally. For example, $500 - 267$ can be calculated as $499 - 266 = 233$. Here the mental strategy is to subtract one from each of the numbers before performing the calculation, jotting down the new numbers as a memory aid.

There are 507 children in a school. A party of 189 is going on a school trip. How many are not going?

If children decide this calculation is too difficult to be done mentally they might be taught to record it using the standard right-to-left decomposition format:

$$\begin{array}{r} \overset{4}{\cancel{5}} \overset{9}{\cancel{0}} \overset{17}{\cancel{7}} \\ \underline{1 \ 8 \ 9} \\ 3 \ 1 \ 8 \end{array}$$

The discussion about this method should focus on establishing a clear understanding of what is going on here. Again, to avoid this becoming a misunderstood routine, with all the well-known potential for error and confusion, it is particularly helpful to discuss the process in terms of base-ten materials and to help children to make clear connections between the manipulation of the materials and the written record. The key point in this discussion will be how to deal with the challenge of '7 - 9' given that there are no tens in the first number. The explanation will involve connecting the written record with the process of exchanging a hundred for ten tens and then one of these tens for ten ones.

Written calculations with decimal notation for money

In year 4, children will begin to develop and refine written methods for calculations with money, using pounds notation with the 'decimal point'. The two digits following the decimal point are generally regarded as representing the number of pence, although they actually represent hundredths of a pound.

(For this reason, children should be encouraged to record ‘two pounds thirty-five’ as £2.35 rather than £2.35p and ‘five pounds ninety’ as £5.90.)

The written methods for calculations which make most sense to children at this stage will therefore take the digits after the point as representing the number of pence.

Two year 4 children have pledges of £3.45 and £3.49 for a charity for doing a sponsored multiplication tables test. How much is this altogether?

Having had plenty of experience of informal and practical manipulation of money, the obvious way for year 4 children to tackle the addition of £3.45 and £3.49 is by using mental strategies, incorporating jottings where necessary. The following might be the kind of written records made to explain their calculation:

$$\begin{array}{l} \text{So} \\ \text{or} \\ \text{So} \end{array} \begin{array}{l} \text{£3.45} + \text{£3.50} \\ \text{£3.45} + \text{£3.49} \\ \text{£3.50} + \text{£3.50} \\ \text{£3.45} + \text{£3.49} \end{array} \begin{array}{l} = \text{£6.95} \\ = \text{£6.94} \\ = \text{£7} \\ = \text{£7} - 5\text{p} - 1\text{p} \\ = \text{£6.94} \end{array}$$

Others might deal with this addition as:

$$\begin{array}{r} \text{£3.45} \\ + \text{£3.49} \\ \hline \text{£6.00} \\ \text{£0.80} \\ \hline \text{£0.14} \\ \hline \text{£6.94} \end{array}$$

The mental calculations required in each column and in the final line should be well within the children’s competence by this stage. Only when they are able to deal with such calculations mentally are they ready to move on to any form of vertical layout for their additions of sums of money. Children must also be confident in writing pence in pounds notation (eg 120p as £1.20, and 14p as £0.14) and vice versa.

If the standard vertical format for this calculation, working from right to left, were used, the written record might look like this:

$$\begin{array}{r} \text{£3.45} + \text{£3.49} \\ \text{£3.45} \\ \hline \text{£3.49} \\ \hline \text{£6.94} \\ \hline 1 \end{array}$$

This is a much more condensed record and a more efficient procedure than the two previous vertical-format methods. It is the efficiency and economy of this method that needs to be stressed. However, children could find it obscures the meaning of the digits either side of the point as representing the number of pounds and the number of

pence. When introducing this format, plenty of time needs to be allowed for discussing and explaining the procedure, perhaps illustrating the process with coins.

Missing digit problems

A useful experience for children to consolidate their developing skills in written recording of additions and subtractions is to find the missing digits in a calculation. This also provides the teacher with an assessment of children's understanding. For example, children might be challenged to write in the missing digits in the following:

$$\begin{array}{r} \square 8 + 27 = 5\square \\ 107 - \square 9 = 58 \end{array} \qquad \begin{array}{r} 12\square \\ + 4\square 6 \\ \hline 500 \\ 70 \\ \hline 12 \\ \hline 582 \end{array} \qquad \begin{array}{r} 3\square 7 \\ - 5\square \\ \hline 273 \end{array}$$

Years 5 and 6

In years 5 and 6, children should continue to use written recording of addition and subtraction in the same three ways as in years 3 and 4: jottings to support those mental calculations they cannot do wholly in their heads; a written record that explains their mental calculations to someone else; and the development of efficient standard written methods for calculations, including column addition and subtraction.

Extending addition

As children work with larger numbers, some way of keeping track of the ones, tens, hundreds, thousands and so on becomes more necessary, so there will be more emphasis on vertical layout.

Use the cards with the digits 2, 3, 4, 5, 6, 7, 8, to make a three-digit and a four-digit number. Find the largest sum of two such numbers that is possible by rearranging these cards.

One possible suggestion is: '8642 + 753'.

There are a number of ways in which this calculation might be recorded, building on the strategy of partitioning into thousands, hundreds, tens and ones, and lining these up vertically in columns. Working from right to left, starting with the ones, leads to recording the above calculation in the standard, condensed form. Some children will still need to make written records of additions in an expanded form, but those who understand and can use the standard algorithm should do so.

$$\begin{array}{r} 8642 + 753 \\ 8000 + 600 + 40 + 2 \\ + \quad 700 + 50 + 3 \\ \hline 8000 + 1300 + 90 + 5 = 9395 \end{array} \qquad \begin{array}{r} 8642 + 753 \\ 600 + 700 \\ 40 + 50 \\ 2 + 3 \\ \hline 8000 \\ 1300 \\ 90 \\ 5 \\ \hline 9395 \end{array}$$

$$\begin{array}{r}
 8642 \\
 + 753 \\
 \hline
 8000 \\
 1300 \\
 90 \\
 5 \\
 \hline
 9395
 \end{array}$$

$$\begin{array}{r}
 8642 \\
 753 \\
 \hline
 9395 \\
 \hline
 1
 \end{array}$$

Extending subtraction

The sides of a sheet of paper are 419mm and 297mm. How much longer is one side than the other?

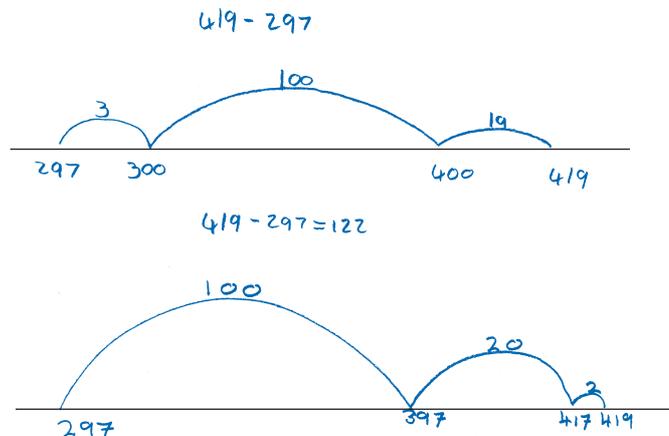
Subtraction in the various contexts of practical measurement is often used for comparing heights, lengths, distances, weights, volumes, capacities, and so on. In this example the difference between the two lengths in millimetres corresponds to the subtraction '419 - 297'.

Using right to left decomposition, start with the units $9 - 7 = 2$; look at the tens next. 90 cannot be subtracted from 10 so exchange 100 for ten 10s, and do $110 - 90 = 20$; then consider the remaining hundreds, $300 - 200 = 100$.

So the answer is $100 + 20 + 2 = 122$. One side is 122mm longer than the other.

$$\begin{array}{r}
 419 \\
 - 297 \\
 \hline
 122
 \end{array}$$

This example can easily be checked mentally by 'adding on'.



The price of a computer system costing £2410 is reduced by 20% (£482). Calculate the new price.

There are now thousands to be considered as well as hundreds, tens and units.

$$2410 - 482$$

$$\begin{array}{r}
 1000 \quad 1300 \quad 100 \quad 10 \\
 2000 + 400 + 10 + 0 \\
 \hline
 400 + 80 + 2 \\
 \hline
 1000 + 900 + 20 + 8 = 1928
 \end{array}$$

which leads to the standard condensed format:

$$\begin{array}{r}
 1 \quad 13 \quad 10 \quad 10 \\
 2 \quad 4 \quad 1 \quad 0 \\
 \hline
 4 \quad 8 \quad 2 \\
 \hline
 1 \quad 9 \quad 2 \quad 8
 \end{array}$$

Addition and subtraction with decimal notation

Children in years 5 and 6 will extend their experience of addition and subtraction into a range of contexts, including calculations with money, and measurement. These contexts provide practical opportunities to extend confidence in calculating with numbers written in decimal notation.

Children need to be competent in calculating mentally additions and subtractions such as $0.3 + 0.6$, $0.5 + 0.8$, $0.06 + 0.04$, $0.008 + 0.009$, $0.9 - 0.7$, $1.2 - 0.8$, $0.10 - 0.07$ and $0.013 - 0.007$ before attempting vertical recording of additions and subtractions with decimals.

An important principle to be established is that to add or subtract two quantities, it is helpful if they are first written in the same units. So, for example, to add 455g to 1.5kg, we have a choice of '455g + 1500g' or '0.455kg + 1.500kg'. Children can be encouraged to switch freely between these choices, depending on the numbers in the problem. Working with mixed units invariably leads to errors in calculation.

In the context of money it will be a well-established practice, by this stage, always to write money in pounds with two figures after the decimal point. It is often helpful to extend this policy to other measuring contexts before any additions or subtractions are done. So, lengths expressed in metres which arise from a measurement in centimetres will be written with two figures after the point (eg 3.50m rather than 3.5m), just like pounds and pence. Lengths arising from measurements in millimetres will be written with three figures after the point (eg 3.500m). Similarly, capacities in litres arising from measurements in millilitres, and mass in kilograms arising from measurements in grams, will be best recorded with three figures after the point, prior to any addition or subtraction (eg 0.750 litres, 1.400kg). As well as enabling the children to retain the meaning of the figures after the point as actual measurements, this procedure helps to ensure that digits and the decimal point are lined up correctly when using a vertical layout.

The capacities of two drinks containers (to the nearest five millilitres) have been measured as 1 litre 665 millilitres and 780 millilitres. How does the combined capacity compare with a container which holds two and a half litres?

The measurements stated in this problem are expressed in a variety of ways. Children will find it helpful to start by writing them either as 1.665 litres, 0.780 litres and 2.500 litres, or as 1665 ml, 780 ml, 2500 ml. The problem will involve first an addition and then a subtraction. Below are some examples of the different ways in which children might record their calculations when working in litres:

1.665 litres + 0.780 litres

$$\begin{array}{r} 1.665 \\ + 0.780 \\ \hline 2.445 \end{array} \quad \text{or} \quad \begin{array}{r} 1.665 \\ + 0.780 \\ \hline 2.445 \\ \hline 1 \end{array}$$

$0.6 + 0.7 = 1.300$
 $0.06 + 0.08 = 0.140$
 0.005
 2.445

The difference in capacities is: 2.500 litres - 2.445 litres = 0.055 litres.
This can be worked out as:

$$\begin{array}{r} 2.500 - 2.445 = 2.499 - 2.444 \\ 2.499 \\ - 2.444 \\ \hline 0.055 \end{array}$$

Adding strings of numbers, including decimals

In years 5 and 6, children will extend their techniques for adding strings of numbers to handle strings of three-digit numbers and decimals. This will involve the extension methods already developed for recording the addition of two numbers,

Find the total weight of five adults weighing 72kg, 57.4kg, 89.75kg, 72.9kg and 89.4kg, to determine if they can all get in a lift with a total weight restriction of 400kg.

Children need to be able to calculate the weight in kilograms as:

$$72.00 + 57.40 + 89.75 + 72.90 + 89.40.$$

Taking the numbers in order, the calculation using the vertical format would look like:

$$\begin{array}{r} 72.00 \\ 57.40 \\ 89.75 \\ 72.90 \\ 89.40 \\ \hline 381.45 \\ 32 \end{array}$$

381.45 kg is less than 400 kg so they are safe in the lift.

Children who are not very confident with the column addition can use an expanded form as before.

When calculations are more complicated than this, it is more efficient and reliable to use a calculator if permitted to do so.

Part 4

Multiplication and division

Introduction

Multiplication and division are introduced whenever objects or numbers are combined or partitioned in equal groups. Although multiplication is usually associated with the idea of repeated addition, it can also involve the idea of a ‘multiplying factor’ such as when one set has ‘four times as many’ elements as another, or one measurement is ‘four times as big’ as another.

Division is associated with repeated subtraction or with sharing equally. Division calculations are often based on knowledge of multiplication facts and so it helps to know that the two operations are linked: division is the inverse of multiplication and vice versa. Children need to be aware that knowledge of one fact can help them derive another. For example if they know $3 \times 4 = 12$ they can use this to work out $6 \times 4 = 24$ or $12 \div 4 = 3$ or $12 \div 3 = 4$.

Written recording of multiplication and division starts in key stage 1 but is developed mainly in key stage 2. This recording takes the form of:

- recorded patterns and sequences of numbers to learn facts and to derive new facts;
- jottings to support mental calculations;
- expanded written recording to show stages in a calculation;
- efficient calculating and recording methods.

Children should be taught to use a wide range of language and the symbols associated with multiplication and division. Written recording should include use of a range of appropriate words, phrases and symbols.

Confusion exists about the way multiplication is written, 6×2 and 2×6 are often interpreted in exactly the same way. It may be argued that the ‘correct’ interpretation for 2×6 is ‘2 multiplied by 6’, that is two taken six times or $2 + 2 + 2 + 2 + 2 + 2$. This fits with the other operations where addition, subtraction and division ‘operate on’ the first number (to the left of the operation sign) with the second (to the right of the operation sign).

This changes if we take the ‘everyday’ interpretation of 2×6 as ‘2 times 6’, or 2 lots of 6, that is six taken two times or $6 + 6$. It is important to be aware that confusion might arise where the expressions are mixed up. Since multiplication obeys the commutative rule ($a \times b = b \times a$), both interpretations are valid and children need to appreciate that 2×6 gives the same result as 6×2 . Both expressions will be used and, later, expressions like $2a$ will be used in algebra as $2 \times a$ meaning ‘2 times a’.

The use of the symbol \div for division can also cause problems. The statement $12 \div 3$ arises out of two practical situations, either as ‘share 12 objects equally among 3 people’ or as ‘how many groups of 3 are there in 12?’ Both are symbolised by $12 \div 3$ which needs to be read as ‘12 divided by 3’. It is important for children to realise that $12 \div 3$ is not the same as $3 \div 12$. Expressions such as ‘12 divided into 3’ are confusing and should be avoided.

In the same way the notation $3\overline{)12}$, although standard for performing short or long division, is confusing because it reverses the order of the numbers compared to the notation $12 \div 3$. It is best to delay the introduction of this notation until late in year 4 when it may be one of the ways to record division calculations.

It is important to introduce a gradient of difficulty for examples using particular methods of multiplication and division. At each stage of development children should reach their full potential. The teacher's task is to help them utilise their knowledge, skills and understanding to maximum effect. Children should be presented with problems ranging in difficulty, that allow them to practise using a common structured method for arriving at the answer. Children also need experience at problem solving in which they have to choose the appropriate method of solution.

Key stage 1

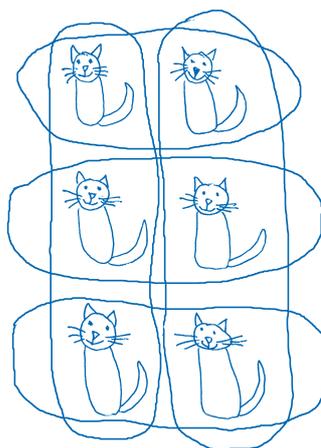
The ideas of multiplication and division begin with counting patterns and contexts involving equal groups, and can be introduced wherever equal groupings are involved. At first, results will be recorded using number patterns and phrases like 'lots of' or 'shared between', but by year 2 children should be introduced to the \times and \div symbols and an appropriate extended vocabulary.

Rhymes and stories which involve counting in twos or in fives ('One, two, three, four five, once I caught a fish alive ...'), or counting forwards or backwards in different intervals from a given starting number, help to develop familiarity with number patterns and sequences. Practical activities, asking questions such as 'How many pencils do I need if everyone has to have 2?' and 'I have 12 pencils to share equally among the three of you; how many will you each have?' and doubling and halving, etc, begin by using actual objects, with the activities recorded using a mixture of pictures, tally marks and symbols.

Pictorial representation

The early recording of experiences of multiplication and division will probably take the form of pictorial representation or a written description. Visual images provide good opportunities to discuss the ways different numbers can be put together and split up, and the discussion can be used to introduce appropriate language.

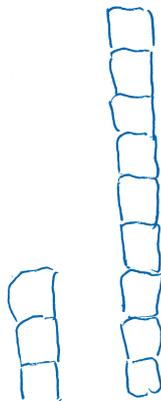
Children can be given pictures of six objects and asked to put rings round to show them as 2 lots of 3 or 3 lots of 2.



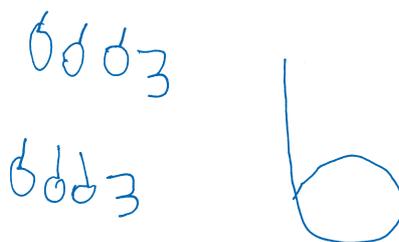
The idea of a **multiplying factor** can be introduced by activities such as:

Make a necklace with red and yellow beads using three red beads for every yellow bead.

Use the bricks to make a tower three times as high as this one:



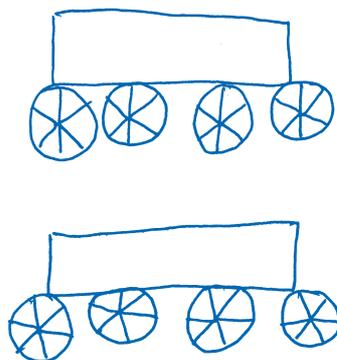
Young children should be familiar with the language of sharing and understand that six being shared equally among three people means everyone has two each and that if they were shared between two people, both would have three:



This is an example of partitioning sets into equal groups and is good preparation for the alternative interpretation of division as repeated subtraction that arises out of activities such as:

How many cars can you make if you have eight wheels? How many motor bikes?

Children can draw pictures to record the two groups of four wheels and talk about grouping eight into two groups of four.



Teachers can discuss the fact that eight can also be grouped as four groups of two, reinforcing the ideas that $8 \div 2 = 4$ and $8 \div 4 = 2$.

Introducing symbols for multiplication and division

Make 12 cakes for the class shop. Arrange them in rows. How many rows are there if there are three cakes in each row? Four in each row? How many groups of six can you make? How many groups of two? What happens when they are arranged five in each row? How many are left over?

Identifying pattern and number statements such as

$$3 + 3 + 3 + 3 = 12; 4 + 4 + 4 = 12; 12 = 6 + 6; 12 = 2 + 2 + 2 + 2 + 2 + 2$$

can introduce the idea of repeated subtraction.

Repeated subtraction to model division is the inverse of repeated addition to model multiplication. It is more helpful to think of division as repeated subtraction than it is to picture division as sharing. Repeated subtraction can also be represented as fixed jumps along a number line. Furthermore, it is a helpful description of the division process. For example, the division $20 \div 2$ can be thought of as 'How many 2s make 20?'

Then, from year 2, the multiplication symbol can be introduced as a succinct way of writing repeated addition and the division symbol for 'sharing' or repeated subtraction:

What is the value of 6 two-pence coins?

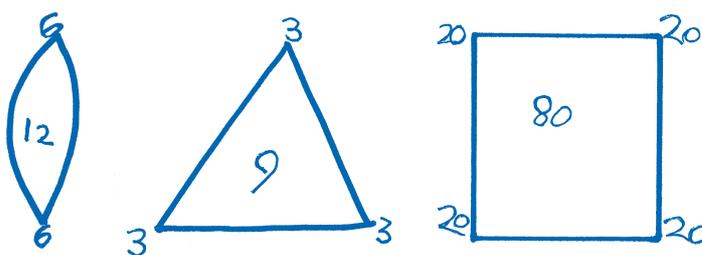
Children can record their working as

$$2 + 2 + 2 + 2 + 2 + 2$$

6 groups of 2

$$2 \times 6.$$

Recording numbers in a variety of spatial arrangements can be used to introduce ideas that can later be extended to multiplication and division.



$$12 = 2 \times 6$$

$$9 = 3 \times 3$$

$$80 = 4 \times 20$$

$$80 \div 4 = 20$$

$$80 \div 20 = 4$$

Years 3 and 4

During years 3 and 4, children will be taught additional facts and will work on different ways to derive new facts from those they already know.

It is important for children to understand, as soon as possible, the commutativity of multiplication (although they do not need to know the name of the rule). This will not only halve the number of facts to be memorised but will help them to decide on the most efficient way to calculate. Children also need to understand that every multiplication fact has two corresponding division facts, for example if they know that $5 \times 3 = 15$ they also know that $15 \div 3 = 5$ and that $15 \div 5 = 3$.

Written recording of related facts will help children to make the connections they will need when calculating.

Written recording will focus on:

- gaining a good understanding of the meanings of the operations and the different ways results can be symbolised;
- recognising that multiplication and division are inverse operations to each other;
- learning multiplication and related division facts;
- making connections between numbers, eg 36 is a multiple of 3 and of 6 and of 9 and of 12; 16 is double 8.
- developing and refining written methods for multiplying and dividing two-digit numbers by a single digit.

Grouping or 'chunking' numbers

When using repeated addition, children can combine pairs or bigger groups of numbers using facts they already know. This grouping is often referred to as 'chunking' and becomes a more economical way of recording.

Sui has 24 plasticine legs to make animals for a display. How many lizards can she make? If she made spiders instead, how many spiders could she make?

Written recording will help children to think of the number 24 as made up either as six groups of 4 or as three groups of 8.

$$4 + 4 + 4 + 4 + 4 + 4 = 24$$

$$6 \times 4 = 24$$

$$8 \times 3 = 24$$

$$4 \times 6 = 24$$

$$3 \times 8 = 24$$

Different ways to record this can be encouraged using both multiplication and division:

$$\begin{array}{ccccccc} 4 & + & 4 & + & 4 & + & 4 & + & 4 & + & 4 \\ & \swarrow & & \searrow & & \swarrow & & \searrow & & \swarrow & & \searrow \\ & & 8 & & 8 & & 8 & & & & & \end{array}$$

6 groups of four can be arranged as 3 groups of eight

$$4 \times 6 = 8 \times 3$$

24 is 6 groups of 4

24 is 3 groups of 8

$$24 \div 4 = 6$$

$$24 \div 8 = 3$$

$$24 \div 6 = 4$$

$$24 \div 3 = 8$$

24 children in teams of 4 make 6 teams.

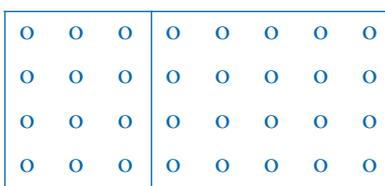
What about 24 children in teams of 8? 48 children in teams of 8? 48 in teams of 16?

Children can record the possible results that can be derived from knowing that $12 \div 4 = 3$ and record some of the results in an ongoing ‘chain’, using doubling or halving:

$$24 \div 4 = 6 \quad \rightarrow \quad \begin{array}{c} \boxed{24 \div 8 = 3} \\ \downarrow \\ \boxed{48 \div 8 = 6} \\ \downarrow \\ \boxed{48 \div 16 = 3} \end{array}$$

Diagrams can help in appreciating how known facts can be used to derive new ones:

Thirty-two pegs on a peg board are to be arranged in fours. Find some different ways to show how many fours, and record your patterns on squared paper.



This illustrates the **distributive rule** (again children do not need to know the name of the rule) that is most easily understood when it is put into words: ‘Eight lots of four is the same as three lots of four together with five lots of four’ or $(8 \times 4) = (3 \times 4) + (5 \times 4) = 12 + 20 = 32$.

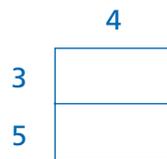
Multiplication is distributed over addition: $a \times (b + c) = (a \times b) + (a \times c)$ for any values of a , b and c . Using the commutativity of multiplication, this statement is equivalent to the statement: $(b + c) \times a = (b \times a) + (c \times a)$.

Different arrangements show different facts. For example, 32 is also the same as 3 lots of 4 and 3 lots of 4 and 2 more lots of 4. An alternative fact from the following array would be $4 \times 8 = 4 \times (3 + 3 + 2) = (4 \times 3) + (4 \times 3) + (4 \times 2)$.

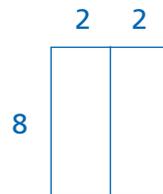
o	o	o	o	o	o	o	o
o	o	o	o	o	o	o	o
o	o	o	o	o	o	o	o
o	o	o	o	o	o	o	o

Another way to record relationships like these is to use the rectangle or grid method:

$$4 \times 8 = 4 \times (3 + 5) = (4 \times 3) + (4 \times 5)$$



$$\text{Alternatively, } 4 \times 8 = (2 + 2) \times 8 = (2 \times 8) + (2 \times 8)$$



In year 4, children will benefit from a lot of experience in ‘breaking down’ calculations in a variety of ways. The distributive rule helps to derive new facts involving increasingly larger numbers, and will be most important in helping children understand and record their calculations when two-digit multipliers are introduced

eg

$$2 \times 14$$

14 times 2 is 10 times 2
plus
4 times 2

$$2 \times 10 = 20$$

so

$$2 \times 14 = 28$$

$$2 \times 4 = 8$$

Recording different ways to calculate will help children understand connections and calculate facts they cannot instantly remember.

Subtraction can help rapid calculation of the multiples of nine, eg $9 \times 8 = (10 \times 8) - (1 \times 8)$. A similar approach provides quick ways to multiply by numbers like 29 (as $30 - 1$).

Seven loaves are needed for sandwiches for sports day. Each loaf costs 29p. How much money is needed?

This could be recorded as

$$\begin{aligned} 7 \times 29 &= 7 \times (30 - 1) = (7 \times 30) - (7 \times 1) \\ 7 \times 30 &= 7 \times (3 \times 10) = (7 \times 3) \times 10 = 210 \\ 210\text{p} - 7\text{p} &= 203\text{p} \end{aligned}$$

Note that multiplication is also distributed over subtraction: $a \times (b - c) = (a \times b) - (a \times c)$ for any values of a , b and c .

Introducing remainders

In years 3 and 4, the ideas of remainders or fractions in the answers to division calculations become more widely used.

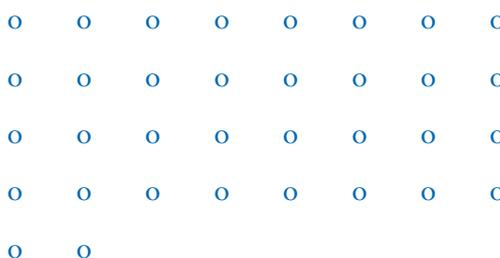
Share 9 cakes between 4 people.

9 cakes shared between 4 people
First find the closest number to 9 in the 4 times table which is $4 \times 2 = 8$ but you're still left with the remainder of 1. Then divide that into four which is a quarter. So each person has 2 and a quarter cakes.

In this context it is possible to share the cake that is left to get the answer 'two and a quarter'.

34 pens are to be shared out equally between eight tables. How many pens on each table?

Associated with the multiplication fact $8 \times 4 = 32$ are the division facts that $32 \div 4 = 8$ and $32 \div 8 = 4$. To solve the division $34 \div 8$ it is necessary to recognise the 34 as $32 + 2$ so that $34 = (4 \times 8) + 2$.



In grouping problems such as this, the 'two left over' cannot be shared exactly. The word **remainder** can be introduced. This can be written as $34 \div 8 = 4$ remainder 2.

When children are secure with the facts of a multiplication 'table', division problems should involve recording facts for 'close' numbers and associated numbers.

How many weeks are in 35 days? Now find how many weeks and days in 34 and 38 days.

$$35 \div 7 = 5 \text{ so } 35 \text{ days} = 5 \text{ weeks}$$

$$34 \div 7 = 4 \text{ remainder } 6 \text{ so } 34 \text{ days} = 4 \text{ weeks and } 6 \text{ days}$$

$$38 \div 7 = 5 \text{ remainder } 3 \text{ so } 38 \text{ days} = 5 \text{ weeks and } 3 \text{ days}$$

Time is needed to investigate the numbers above and below exact multiples, and written recording will help to establish the ways remainders are calculated in a range of different contexts.

At this stage the purpose is to connect known multiplication and division facts to new results.

Using known multiples, such as those of 6 – 6, 12, 18, 30, etc – children should be encouraged to use the fact that the numbers in between, when divided by 6, will leave a remainder between 1 and 5:

25 is one more than 24 and so $25 \div 6 = 4$ with 1 left over, or 4 remainder 1;
26 is two more than 24 and so $26 \div 6 = 4$ with 2 left over, or 4 remainder 2.

This approach avoids the common error of children recording $29 \div 6$ as 5 remainder 1 instead of 4 remainder 5.

The ideas in this section can be reinforced with missing number problems, for example $37 = (7 \times 5) + \square$ and $(8 \times \square) + 6 = 38$. These problems can also be considered as simple problems in algebra.

Using multiples and powers of ten

Multiplying by ten is one of the basic skills that is required for working with larger numbers; multiplication by multiples of ten (20, 60 ...) and of powers of 10 (100, 1000 ...) follow from it.

It is most important that multiplying by ten or one hundred is not thought of as a case of ‘adding noughts’: apart from being a very inappropriate expression because adding zero leaves a number unchanged, the rule also fails when, for example, 0.2 is multiplied by ten.

The array

4 4 4 4 4 4 4 4 4 4
4 4 4 4 4 4 4 4 4 4
4 4 4 4 4 4 4 4 4 4

can be seen as 4×30 or $4 \times 10 \times 3$ or $4 \times 3 \times 10$. Multiplication by 30 can be calculated by first multiplying by 3 and then by 10.

When three numbers are to be multiplied, **the associative rule** (again, children should understand the importance of this rule, not its technical name) makes it possible to choose pairs in different ways:

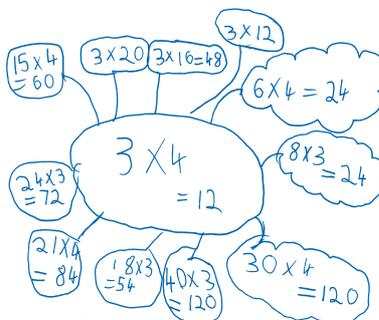
$$4 \times (3 \times 10) = (4 \times 3) \times 10$$

The associative rule together with the commutative rule allows numbers to be multiplied in any order.

$$\begin{aligned} \text{eg } 4 \times 30 &= 4 \times (3 \times 10) \\ &= (4 \times 3) \times 10 \text{ or } 12 \times 10 \\ &= (4 \times 10) \times 3 = 40 \times 3 \end{aligned}$$

Results such as the fact that '4 lots of 30' gives the same total as '3 lots of 40' are not obvious and much practice in making these connections will be necessary.

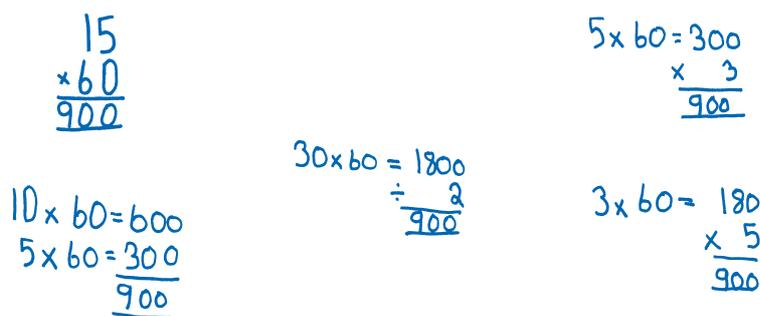
Three bags of marbles with 4 in each gives a total of 12 marbles. What if there were 40 in each bag? What if there were 30 bags of 4? What other multiplication calculations can you work out from $3 \times 4 = 12$?



Informal jottings

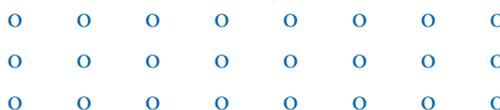
As children begin to work with larger numbers they will need to use personal jottings to help in breaking down a calculation into smaller steps. These personal jottings will help teachers to understand the children's way of working and show whether they are using the most appropriate ways to calculate. Informal jottings will help children keep track of their mental calculations, explore different connections and develop a range of approaches to multiplying and dividing. With support and guidance these jottings will gradually make more use of mathematical symbols and will increasingly be understood by others.

A tyrannosaurus rex was approximately 60 times as long as a lizard. A lizard's tail is 15 cm long. About how long was the tail of a T-rex?



Choosing the facts that are relevant and using mental strategies like doubling and halving are important priorities in years 3 and 4. The teacher's role is to help children find efficient ways to calculate and to record their working in a way that others can understand.

Written recording involving rectangular arrays can help children to see that there are two ways of making equal groups: one views the numbers in rows, while the other views the numbers in columns. The array



can be seen as

$$24 = 8 + 8 + 8 = 8 \times 3$$

$$24 = 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 = 3 \times 8$$

3 times 8 is 24

3 multiplied by 8 is 24

8 multiplied by 3 is 24

8 times 3 is 24

It is important to establish the connections between a number and its factors in a flexible way that allows the most effective interpretations of multiplication and division to be used.

*If 24 children are grouped into teams with 3 in each team, how many teams will there be?
24 children are grouped into teams of 8. How many teams will there be?*

Children can be encouraged to use multiplication and division symbols and associated vocabulary to write the results in different ways.

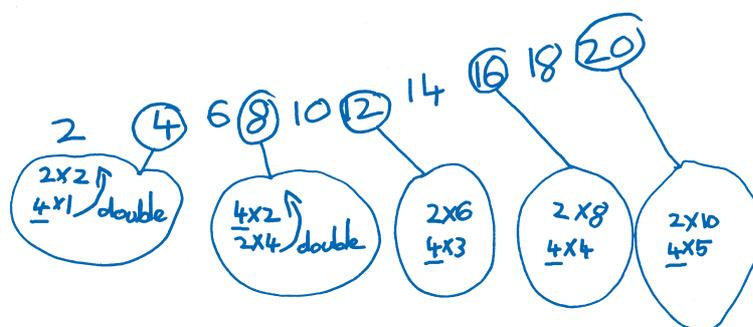
24 has 3 and 8 as factors

24 is the product of 3 and 8

$$3 \times 8 = 24 \quad 8 \times 3 = 24 \quad 24 \div 3 = 8 \quad 24 \div 8 = 3$$

Written recording will help to establish patterns of numbers and to derive new multiplication and division facts.

Write the multiples of 2 and circle every other number. Which number is a factor of all the circled numbers?



These and other patterns can be seen on a 100 square when different colours are used to highlight different multiples.

Multiplication of two-digit numbers by a single-digit multiplier

Initially, work with larger numbers will continue to involve children's own informal strategies that use the facts and rules they already know. Personal jottings will continue to be important as children extend their strategies for working with larger numbers. Multiplication of larger numbers involves 'splitting' the calculation into smaller parts. Repeated addition will become inefficient as the numbers get larger, but this idea can be retained if children move progressively to more efficient 'chunks'. Doubling can also be a helpful strategy. Partitioning using place value is effective and forms a basis for future more compact methods. Written recording should express clearly the strategy used and children will need help to develop an organised format for recording, using mathematical symbols.

The rectangle or grid method can be used to introduce partitioning using place value.

How many sweets are needed for party bags if 27 children are to have 6 each?

$$\begin{array}{r} \times \quad 20 \quad 7 \\ 6 \quad \boxed{120} \quad \boxed{42} \end{array} = 162$$

This may lead to an expanded form of vertical recording, although by this stage some children will be able to use a more compact recording.

$$\begin{array}{r} \times \quad 27 \\ 20 \times 6 \quad 120 \\ 7 \times 6 \quad 42 \\ \hline 162 \end{array} \qquad \begin{array}{r} \times \quad 27 \\ \quad 6 \\ \hline 162 \\ \quad 4 \end{array}$$

Division of two-digit numbers by a single-digit divisor

When the numbers to be multiplied go beyond the 10×10 multiplication facts, children will have to decide which facts they need to use and which methods will be the most efficient.

96 pears are to be sold in packets of 4. How many packets will there be?

It is generally helpful to identify division with repeated subtraction, eg $96 \div 4$ can be read as ‘how many fours will make 96?’. This can be calculated by ‘chunking’ the 96 into numbers that are readily associated with four, such as $96 = 40 + 40 + 16$. Listing results vertically will help lead to a standard written method. When work is set out in columns it is important that units are lined up under other units, tens under tens and so on.

$96 = 40 + 40 + 16$		96
$= (10 \times 4) + (10 \times 4) + (4 \times 4)$		$- 40$
$= 24 \times 4$		56
		$- 40$
		16
		$- 16$
		0
		10 packs
		10 packs
		4 packs
		24 packs

This recognition of related ‘chunks’ will often involve multiples of ten and provides a strategy that will extend to a division of any size. Just as multiplication ‘builds up’ numbers in stages, division can be seen as ‘breaking down’ numbers. This emphasises that division is the inverse operation to multiplication.

A shop notice states that there are 87 shopping days to Christmas. How many weeks is that?

$$\begin{array}{r} 7 \overline{) 87} \\ - 70 \\ \hline 17 \\ - 14 \\ \hline 3 \end{array} \quad \begin{array}{l} 10 \times 7 \\ 2 \times 7 \end{array}$$

Answer: 12 weeks and 3 days

Years 5 and 6

In years 5 and 6, informal written calculations for multiplication and division will depend on the numbers involved and the ways they are connected. Standard written methods of multiplication and division will be developed progressively, but it is most important that at all stages children understand the methods they are using. Some children will need to use an expanded written form of calculation for longer than others. Children should check their answers, preferably using a different method, for example the inverse operation. The process of identifying the mathematical rules that are involved in these calculations will help them to recognise how the rules of number are applied in algebra. This will be studied in key stage 3.

The written recording in years 5 and 6 will focus on:

- making appropriate choices from a flexible range of strategies;
- relating known facts to larger numbers and to decimals;
- establishing clear and efficient ways to record working, moving towards 'long multiplication' and 'short division' (able children could progress to recording 'long division').

Not all written work will involve larger numbers, as smaller numbers continue to be used to reinforce number relationships and rules that help when calculating.

Multiplication with larger numbers and a single-digit multiplier

The class wants to make 275 spiders for a display. How many legs do they need to make?

Some children may do this mentally, supported by jottings:

$$275 \times 10 = 2750$$

$$275 \times 2 = 550$$

$$2750 - 550 = 2200$$

$$300 \times 8 = 2400$$

$$25 \times 8 = 200$$

$$2400 - 200 = 2200$$

275 doubled is 550

550 doubled is 1100

1100 doubled is 2200

A grid method (or area method) might be used which emphasises the numbers as a whole rather than individual digits:

	200	70	5
8	1600	560	40

Some will work towards a conventional short multiplication, using an expanded layout:

$$\begin{array}{r} 275 \\ \times 8 \\ \hline 1600 \\ 560 \\ \hline 40 \\ \hline 2200 \end{array}$$

$$\begin{array}{r} 275 \\ \times 8 \\ \hline 40 \\ 560 \\ \hline 1600 \\ \hline 2200 \end{array}$$

The traditional ‘short multiplication’ is a highly efficient way to calculate, but it has a very condensed form and needs to be introduced carefully. Each time a multiplication of two single digits gives a number greater than 9 (eg $8 \times 5 = 40$), the ‘tens part’ must be ‘carried forward’ and added to the result of the next calculation (ie in the example below, 4 must be added to the result of $8 \times 7 = 56$). The calculation is set out as shown:

$$\begin{array}{r} 275 \\ \times 8 \\ \hline 2200 \end{array}$$

It is important that the ‘traditional’ standard written method is seen as the ultimate objective. However, children will need to use an expanded form for such calculations before the steps in the method are meaningful and they can use this efficient method with understanding.

Multiplication with two- and three-digit numbers

Children should progress to multiplication of two- and three-digit numbers by two-digit numbers using the same vertical recording methods they have established for single-digit multiplication.

For the school fete, 24 children have each made 16 cakes. How many cakes altogether?

Although the context suggests 24 groups of 16, 24×16 can be calculated as 24 sixteens or as 16 twenty-fours.

$$\begin{array}{l} 24 \times 16 \left\{ \begin{array}{l} 24 \times 10 = 240 \\ 24 \times 6 = 120 + 24 \end{array} \right. \\ \qquad \qquad \qquad 240 + 120 + 24 = 384 \end{array}$$

or

$$\begin{array}{l} 24 \times 16 \left\{ \begin{array}{l} 20 \times 16 = 320 \\ 4 \times 16 = 64 \end{array} \right. \\ \qquad \qquad \qquad 320 + 64 = 384 \end{array}$$

Special methods can be introduced for multiplication by 25, which can be considered as division by four followed by multiplication by 100. For example, $36 \times 25 = (36 \div 4) \times 100 = 9 \times 100 = 900$.

It is often useful to factorise a multiplier to carry out a multiplication. Thus, for example:

$$\begin{aligned} 36 \times 42 &= 36 \times 6 \times 7 \\ &= 216 \times 7 \\ &= 1512 \end{aligned}$$

When multiplying a three-digit number by a two-digit number, the different stages of the calculation need to be recorded in an organised way and a vertical arrangement is appropriate, although some children may still find a grid method easier to understand. Starting with 'long hand' notes explaining the method, children should be guided to more efficient ways of recording.

As before a grid method can help understanding

	20	4
10	200	40
6	120	24

$$24 \times 16 = 200 + 120 + 40 + 24 = 384$$

Children might use any of the stages that lead to the standard written method:

$\begin{array}{r} 24 \\ \times 16 \\ \hline 24 \\ 40 \\ 120 \\ 200 \\ \hline 384 \end{array}$	$\begin{array}{r} 24 \\ \times 16 \\ \hline 240 \\ 144 \\ \hline 384 \end{array}$	$\begin{array}{r} 24 \\ \times 16 \\ \hline 240 \\ 144 \\ \hline 384 \end{array}$
4×6 4×10 20×6 20×10	24×10 24×6	

How many hours are there in 1 year?

Using a grid method:

	300	60	5
20	6000	1200	100
4	1200	240	20

$$365 \times 24 = 6000 + 1200 + 100 + 1200 + 240 + 20 = 8760$$

or

or

or

$\begin{array}{r} 365 \\ \times 24 \\ \hline 100 \\ 1200 \\ 6000 \\ 20 \\ 240 \\ 1200 \\ \hline 8760 \end{array}$	$\begin{array}{r} 365 \\ \times 24 \\ \hline 7300 \\ 1460 \\ \hline 8760 \end{array}$	$\begin{array}{r} 365 \\ \times 24 \\ \hline 7300 \\ 1460 \\ \hline 8760 \end{array}$
5×20 60×20 300×20 5×4 60×4 300×4	365×20 365×4	

A harder example, using the grid method, is to calculate 203×57 . This is set out as follows:

	200	0	3
50	10000	0	150
7	1400	0	21

$10000 + 1400 + 150 + 21 = 11571$

The expanded column form is set out as:

$$\begin{array}{r}
 203 \\
 \times \quad 57 \\
 \hline
 10150 \quad 203 \times 50 \\
 \underline{1421} \quad 203 \times 7 \\
 11571
 \end{array}$$

The compact form omits the prompts.

Division with larger numbers and a single-digit divisor

Estimation skills are needed for division with larger numbers in order to make a sensible choice of multiple to subtract. Approximating first helps children decide if the results of their calculations are reasonable.

458 stickers are shared equally between three children. How many does each get?

Since $100 \times 3 = 300$ and $200 \times 3 = 600$, the answer is between 100 and 200.

Chunking can be used to perform the division:

$$\begin{array}{r}
 458 \\
 - \underline{300} \quad 100 \times 3 \\
 158 \\
 - \underline{150} \quad 50 \times 3 \\
 8 \\
 - \underline{6} \quad 2 \times 3 \\
 2
 \end{array}$$

Answer: 152 remainder 2.

The ‘traditional’ written division method presents a very contracted format for recording. In the case of $458 \div 3$, it would look like:

$$\begin{array}{r}
 152 \text{ rem } 2 \\
 3 \overline{)458}
 \end{array}$$

To use this format, children will need to be reminded that the 4 means four hundred and when this is divided by 3 the result is in hundreds. In this case, 3 ‘goes into’ 4 once with 1 left over. That means there is one hundred in the answer and the remaining one hundred needs to be exchanged for 10 tens. There are now 15 tens to divide by 3, giving 5 tens as the result. Next, 8 units divided by 3 gives 2 units with 2 left over and the answer is 152 remainder 2.

Another feature of this method which needs careful explanation is the procedure which arises when zeros need to be included in the answer. It is a common error for them to be omitted altogether.

In Barton zoo a week's supply of 1256 apples is to be shared equally between 6 elephants. How many apples will each elephant get?

$$\begin{array}{r}
 209 \\
 6 \overline{)1256} \\
 \underline{600} \quad \times 100 \\
 656 \\
 \underline{600} \quad \times 100 \\
 56 \\
 \underline{54} \quad \times 9 \\
 2
 \end{array}$$

$$\begin{aligned}
 1256 &= 1200 + 56 \\
 1200 &= 200 \times 6 \\
 56 &= 54 + 2 \\
 54 &= 9 \times 6
 \end{aligned}$$

Answer 209 with 2 left

Answer 209 rem 2

The compact short division is:

$$\begin{array}{r}
 209 \text{ rem } 2 \\
 6 \overline{)1'25'6}
 \end{array}$$

In this example, the digits need to be carefully aligned in the appropriate columns. An advantage of a strategy based on repeated subtraction of 'chunks' is that the same method can be extended to work with two-digit divisors.

Division with two-digit divisors

To divide by a two-digit divisor, children will need to select the most effective approach for each task they meet, including whether to use the standard 'long division' algorithm. Some calculations will be best undertaken with a calculator.

A collection of 128 pencils is to be packed equally into 16 boxes. How many pencils in each box?

Standard 'long division' does not help here. Approaches based on repeated subtraction will always work, and the choice of appropriate chunks can make the calculation more efficient. The facts that children know about 16 may lead to a doubling or halving approach to the problem. Approximation suggests that the answer should be a bit more than 6, since $120 \div 20 = 6$.

$$\begin{array}{r}
 128 \quad 8 \times 8 = 64 \quad \text{half 128 is 64} \\
 \underline{-32} \quad 64 \div 8 = 8 \quad \text{half 64 is 32} \\
 96 \quad 128 \div 8 = 16 \quad 2 \times 16 = 32 \\
 \underline{-32} \quad 128 \div 16 = 8 \quad 4 \times 16 = 64 \\
 64 \quad 8 \times 16 = 128 \\
 \underline{-32} \\
 32 \\
 \underline{-32} \\
 0
 \end{array}$$

432 school children are going on an outing. If each bus takes 15 passengers, how many buses will be needed?

The setting of problems in 'real' contexts has the advantage that the context may suggest a way of calculating the answer. Written recording will represent the mental approach used, and will help keep track of stages in the calculation. Children should continue to compare their approaches and discuss appropriate ways to write them

down. They could start, for example, by suggesting that the answer lies between 20 and 40, since $400 \div 10 = 40$ and $400 \div 20 = 20$. Then the calculation might be recorded as:

$$\begin{array}{r}
 432 \\
 10 \text{ buses } \quad -150 \\
 \hline
 282 \\
 10 \text{ buses } \quad -150 \\
 \hline
 132 \\
 4 \text{ buses } \quad -60 \\
 \hline
 72 \\
 4 \text{ buses } \quad -60 \\
 \hline
 12
 \end{array}$$

Answer 28 remainder 12. Add 1 more bus = 29

$$432 = 300 + 132$$

$$132 = 120 + 12$$

$$\text{So } 432 = 300 + 120 + 12$$

$$20 \text{ buses and } 8 \text{ buses and } 1 \text{ more} = 29 \text{ buses}$$

$$\text{or as } 432 = 450 - 18$$

450 would need 30 buses
18 fewer

So 29 buses are needed.

These methods record the thinking that has been used. The standard written method can be set out in expanded form as

$$\begin{array}{r}
 28 \\
 15 \overline{)432} \\
 \underline{300} \quad \times 20 \\
 132 \\
 \underline{120} \quad \times 8 \\
 12 \quad \times 28
 \end{array}$$

Answer 28 remainder 12.

or, succinctly, as

$$\begin{array}{r}
 28 \\
 15 \overline{)432} \\
 \underline{300} \\
 132 \\
 \underline{120} \\
 12
 \end{array}$$

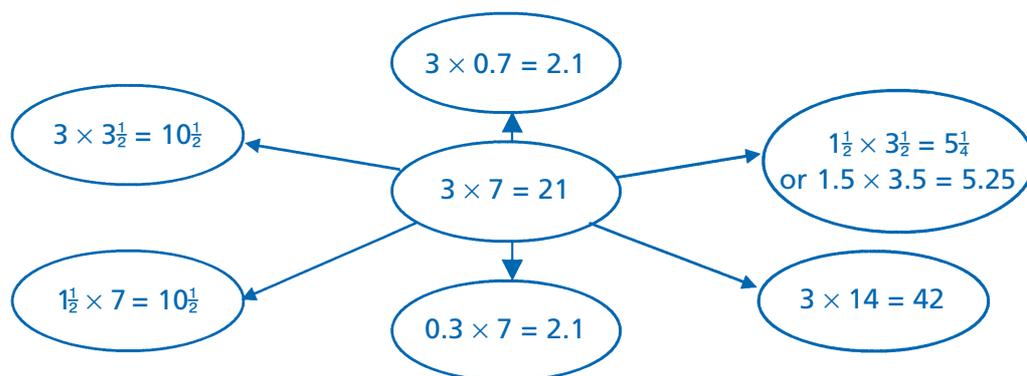
Answer 28 remainder 12.

Division is emphasised through its inverse operation, multiplication. The calculation involves working successively through the hundreds, tens and units columns. In this case, the largest tens multiple of 15 that can be used is 20. Since $15 \times 20 = 300$, another 132 still has to be accounted for ($432 - 300$). The largest single-digit multiple of 15 which does not exceed 132 is 8; $15 \times 8 = 120$ leaving 12 ($132 - 120$) still not accounted for. The 12 cannot be further sub-divided without using decimals or fractions. The correct alignment of digits is critical in carrying out this method successfully.

Different contexts, particularly those that relate to the children's own experiences, are useful for generating multiplication and division calculations, for example in working out the cost per portion of different cereals, or the costing for a class outing. These will incorporate the important steps of formulating the problem and interpreting the answer in a meaningful way as well as identifying appropriate solution strategies. For example, in the context above, the answer '28 remainder 12' is not appropriate as 29 buses will be needed. Collaboration on solving such problems may also generate comparisons between mental, written and calculator strategies.

Multiplication with decimals

Multiplication of fractions and decimals will be related to whole number multiplication. For example:



In year 5, multiplication calculations will be extended to include multiplying numbers with one decimal place by a single digit, and in year 6, numbers with two decimal places by a single digit.

A chicken's egg is 5.4cm long. If a dodo's egg was 4 times as long, how long would that be?

5.4×4	
$5.0 \times 4 = 20.0$	$5.4 \times 2 = 10.8$
$0.4 \times 4 = 1.6$	$10.8 \times 2 = (10 \times 2) + (0.8 \times 2)$
21.6	$= 20 + 1.6$
Dodo's egg is 21.6cm long	$= 21.6$
	Dodo's egg is 21.6cm long

Money provides good examples of numbers with two decimal places.

A kite costs £23.75. How much would it cost to buy three of these kites?

The calculation should be done mentally as

$$\begin{aligned} \pounds 24 \times 3 - \pounds 0.25 \times 3 &= \pounds 72 - \pounds 0.75 \\ &= \pounds 71.25 \end{aligned}$$

or it could be performed as

$$\begin{aligned} \pounds 23.75 \times 3 &= \pounds 2375 \times 3 \div 100 = \pounds 7125 \div 100 \\ &= \pounds 71.25 \end{aligned}$$

or as

23.75×3	$23 \times 3 = 69.00$
	$0.7 \times 3 = 2.10$
	$0.05 \times 3 = \underline{0.15}$
	71.25

It would cost £71.25.

Children who are used to the traditional written method for multiplication could use the same layout for numbers with decimals and a whole number multiplier

$$\begin{array}{r} 23.75 \\ \times \quad 3 \\ \hline 71.25 \end{array}$$

The grid method can be adapted to multiply numbers with decimal points. For example, 23.1×1.7 could be calculated as follows:

	20	3	0.1
1	20	3	0.1
0.7	14	2.1	0.07

$$\begin{aligned} \text{Hence } 23.1 \times 1.7 &= 20 + 14 + 3 + 2.1 + 0.1 + 0.07 \\ &= 39.27 \end{aligned}$$

The calculation can also be performed as $231 \times 17 \div 100$. Using column multiplication, this gives for 231×17 :

$$\begin{array}{r} 231 \\ \times 17 \\ \hline 1617 \\ 2310 \\ \hline 3927 \end{array}$$

Hence $23.1 \times 1.7 = 39.27$, reintroducing the decimal point by dividing 3927 by 100.

Division with decimals

Decimal numbers may occur when measurements are taken and when division calculations are related to real situations.

A car parking space is 23.4 metres long. If 9 cars are to be parked side by side, what width would each parking bay be?

An approximate answer would be ‘about 2 and a half metres’:

$$23.4 \div 9$$

My estimation is 2.5 metres because I rounded up 23.4 to 25 and 9 to 10. $25 \div 10 = 2.5$

A written method for this calculation can be built on procedures for whole numbers.

$$\begin{array}{r} 9 \overline{) 23.4} \\ \underline{-18.0} \quad 2 \times 9 \\ 5.4 \\ \underline{-4.5} \quad 0.5 \times 9 \\ 0.9 \\ \underline{-0.9} \quad 0.1 \times 9 \\ 0.0 \quad 2.6 \times 9 \end{array} \qquad 23.4 \div 9 = 2.6$$

Unfortunately, not all calculations will work out as exactly as this one, and care needs to be taken in choosing the first examples children will meet.

Part 5:

Fractions, decimals and percentages

Introduction

Children are expected to be familiar with the concept of fractions and decimals in the context of money in key stage 1, and to know the relationship between simple fractions, decimals and percentages in key stage 2. Some calculations will be carried out mentally and children will record their working informally, again with increasing efficiency. This chapter discusses the range of such written recordings that children might use to support and explain their mental strategies.

Reception and key stage 1

A variety of ways of recording fractions should be introduced, including pictures, diagrams, words and symbols. Through activities such as folding in half, making a half turn, knowing the half hour, and sharing equally, children will become familiar with the idea of halves and quarters. They will be able to find $\frac{1}{2}$ and $\frac{1}{4}$ of an object or of a group of objects and to begin to recognise that two halves or four quarters of an object make a whole and that two quarters is the same as one half. They are likely to encounter the idea of a fraction in everyday situations such as 'half the class should go to the hall' or 'you can have half each'. By recording the outcome of these activities they can begin to make connections such as:

Two people share a cake, so they have half each; these halves are equal and two halves make a whole.

Four people share a packet of 12 sweets; they have a quarter of them each; each person gets 3 sweets; a quarter of 12 is 3.

The activities that children should be given in order to develop the idea of fractions include:

- halving or quartering a regular shape or object and talking about the outcome;
- halving or quartering a quantity such as a length or weight;
- halving or quartering a discrete collection of objects and relating this to the idea of division.

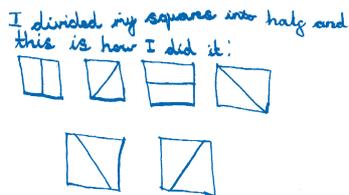
Using objects

Children should be given a variety of shapes and objects to be cut or folded in half. The initial record of the activity will be the actual folded shapes or cut objects. A more permanent record can be made by asking children to draw a picture of what they did:

Cut the banana in half. Can you find another way of doing it?

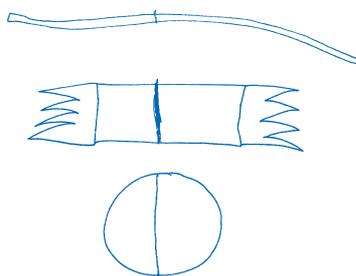


Cut the square in half. In how many different ways can you do it?



Using quantities

In order to find half or a quarter of a continuous quantity, some comparisons will need to be made, either by eye or by folding, in the case of a piece of string, or by balancing the weights of the two halves if it is a piece of plasticine or a small bag of sand. Again, the recording can be pictures of what the children did or can be an account in words which the teacher can scribe for the child:



Using discrete numbers of objects

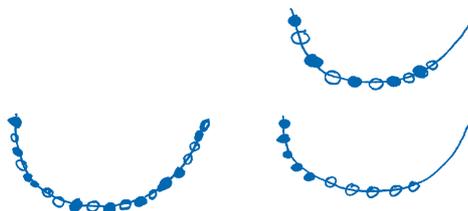
Children need a variety of activities that involve finding a fraction of a discrete set of objects. At first this will be done by sorting the objects into two groups and finding how many there are in each:

Find half of these 20 counters.

Attention can be drawn to the fact that there are ten in each group, that two tens are twenty and so half of twenty is ten. Children can be asked to think about what would happen if 21 counters were used.

Make a necklace that has half its beads red and half white. Find some different patterns with the same number of beads.

Anna arranged 10 beads and then 20 beads, where half were red and half were white:



For her 10-bead necklaces, she said, 'There are five red beads and five black beads each time.' Her teacher emphasised that two fives are ten, so half of ten is 5.

There are sausages for tea and each of four children must be given the same number of sausages. There are 10 sausages altogether. How many does each child get?

One child explained, 'I gave everyone two sausages, then cut each of the other two into half and gave everyone another half sausage.'

Another child wrote:

Half of 10 is 5 and half of 5 is 2 and a half.

Introduction of symbols

Teachers can begin to introduce formal recording of fractions by helping children record their mental calculations:

There are 28 children in the class. A quarter of them have gone swimming. How many children is that?

'I said half of 28 is 14, and half of 14 is 7.'

The teacher can then give the child the written statements:

$$\frac{1}{2} \text{ of } 28 = 14$$

$$\frac{1}{2} \text{ of } 14 = 7$$

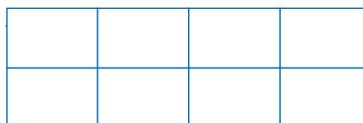
$$\frac{1}{4} \text{ of } 28 = 7$$

Children will initially view the symbols $\frac{1}{2}$ and $\frac{1}{4}$ in their entirety, and will later learn to appreciate the significance of the numerator and the denominator.

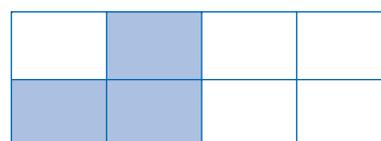
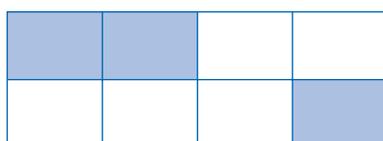
Years 3 and 4

Children in years 3 and 4 will become familiar with a wider range of unit fractions, with fractions which are several parts of a whole, and with mixed fractions. They will begin to use the symbols for recording fractions and record decimal place value in the context of number. Children will continue to need practice at finding fractions of shapes, measures and numbers. The part-whole model provides its own recording device, so teachers can easily see whether children have understood the idea of a fraction in this context:

Find different ways of showing $\frac{3}{8}$ of this rectangle?:



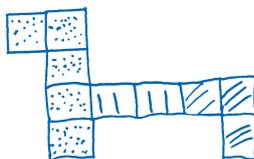
Children should know that any three of the eight squares can be highlighted. These examples both illustrate $\frac{3}{8}$ of the original rectangle.



Children should also be encouraged to use a variety of shapes and measures, such as money or length, so that fractions are not always associated with rectangles or circles:

Use interlocking cubes to make a dinosaur like this that is $\frac{1}{2}$ red, $\frac{1}{5}$ blue and the rest green.

What fraction is green?



I did this by counting the number of squares and counted the green ones and the fraction came to $\frac{3}{10}$ because 3 out of the 10 squares were green.

Sunil saved £12.40 and spends a quarter of it on a book. How much does he spend?

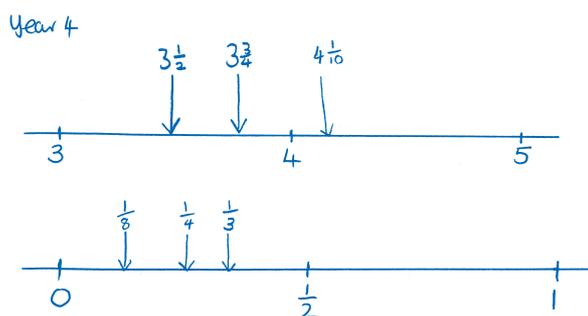
Sunil has £12.40
He wants to spend $\frac{1}{4}$ of it.
To find a quarter, divide by 4

$$12 \div 4 = \text{£ } 3$$

40p \div 4 = 10p so it is £3.10 in the end.

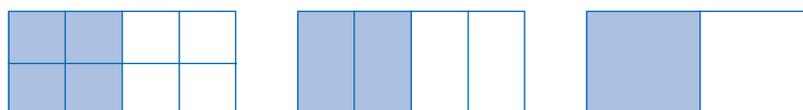
Using a number line

It is important that children begin to think of fractions as numbers, not just as parts of whole shapes, and that these numbers can be placed on a number line. They need to know that the fraction $\frac{1}{2}$ can be placed halfway between 0 and 1 and the number $6\frac{1}{4}$ lies halfway between 6 and $6\frac{1}{2}$. Number lines help children to understand the relationships between fractions, such as $\frac{1}{4}$ is less than $\frac{1}{3}$ because $\frac{1}{3}$ is closer to 1.



Equivalent fractions and ordering fractions

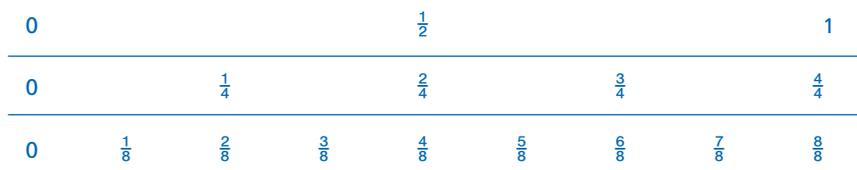
The part-whole model of fractions can be used to establish simple equivalences between fractions.



Three children each have a bar of chocolate the same size and eat half of it

Children can use appropriate symbols to record this as $\frac{4}{8} = \frac{2}{4} = \frac{1}{2}$.

Using a series of number lines, one under the other, sometimes called a fraction wall, can also help to reinforce the idea of equivalence:

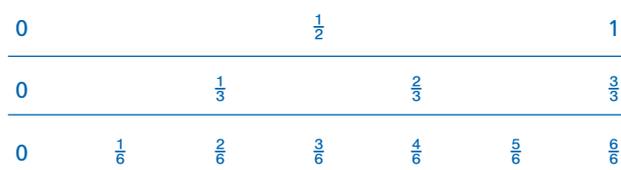


By looking at the fractions that line up vertically, children can see equivalences:

‘I noticed that $\frac{2}{8} = \frac{1}{4}$, $\frac{4}{8} = \frac{2}{4} = \frac{1}{2}$, $\frac{6}{8} = \frac{3}{4}$ and $\frac{8}{8} = \frac{4}{4} = 1$.’

The lines enable the relative sizes of fractions to be seen. For example, $\frac{5}{8} > \frac{1}{2}$ and $\frac{3}{8} < \frac{1}{2}$

Suitable choices of lines can introduce other equivalents, for example:



giving $\frac{2}{6} = \frac{1}{3}$, $\frac{3}{6} = \frac{1}{2}$ and $\frac{4}{6} = \frac{2}{3}$ and $\frac{6}{6} = 1$ and also, for example, the inequalities $\frac{5}{6} > \frac{2}{3}$ and $\frac{1}{6} < \frac{1}{3}$.

Children should now be in a position to find simple fractions by numerical calculation, and examples such as:

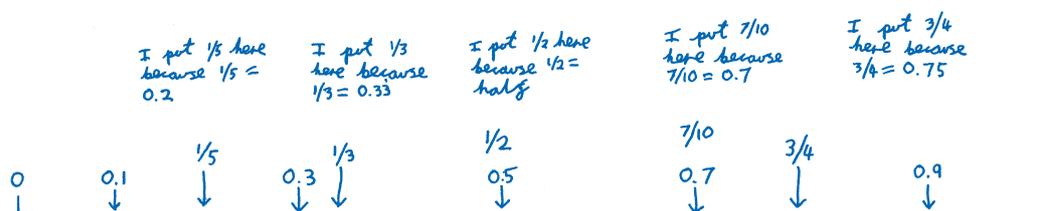
Half of 24 is ?	$\frac{1}{2}$ of ? is 4	Find a third of 12	Find $\frac{1}{10}$ of 20
Half of 12 is ?	$\frac{1}{2}$ of ? is 5	Find a quarter of 16	Find $\frac{1}{10}$ of 70
Half of 9 is ?	$\frac{1}{2}$ of ? is $3\frac{1}{2}$	One fifth of ? is 12	Find $\frac{1}{10}$ of 1
Half of 15 is ?	$\frac{1}{2}$ of ? is $5\frac{1}{2}$	One fifth of ? is 10	Find $\frac{1}{10}$ of 6

enable children to become familiar with recording fractions both in words and in symbols.

Decimal equivalents

Familiar decimals can be linked with corresponding fractions. By knowing, for example, that 0.5 is the same as $\frac{1}{2}$, 0.75 as $\frac{3}{4}$ and 0.3 as $\frac{3}{10}$, children can be asked to put the decimals and the fractions they know on the same number line:

Put the decimals 0.3, 0.1, 0.9, 0.5, 0.7 in order on a 0 to 1 number line. Now put the fractions $\frac{1}{2}$, $\frac{3}{4}$, $\frac{1}{5}$, and $\frac{7}{10}$ in place on the same line.



Children need to be helped to see that $10 \div 4$ can be recorded either as $2\frac{1}{2}$ or as 2.5.

Record the answers you get when 12 is divided by different numbers.

I got whole numbers for $12 \div 2, 12 \div 3, 12 \div 4, 12 \div 6$

For $12 \div 5, 12 \div 7, 12 \div 8, 12 \div 9, 12 \div 10$ and $12 \div 11$ I used a calculator.

$12 \div 5, 12 \div 8$ and $12 \div 10$ gave quite short answers (2.4, 1.5 and 1.2) but $12 \div 7$ was interesting - 1.7142857

Using fractions and decimals for representing data

Fractions and decimals can both be used to represent data that has been collected. Collecting data and representing it in a variety of ways can help to show how the ideas are connected. For example, children can collect information about favourite sports, animals, etc from, say, 20 people and represent their findings. They use data handling software packages to display pie charts and can label each part as a fraction or as a decimal.

Years 5 and 6

Fractions and division

Children need to know that the fraction notation can be interpreted in two ways. For example, $\frac{3}{4}$ can be thought of as a 'whole' divided into four equal parts of which three are taken, or, alternatively, as the result of dividing 3 by 4. The second interpretation, that fractions can be thought of as division, becomes increasingly important for later work with fractions and with algebra. It arises from activities such as 'share three pizzas between four people'. The process is $3 \div 4$ and the outcome is that everyone gets $\frac{3}{4}$ of a pizza. It is important that children realise that both of these essentially different activities are recorded using the same notation, $\frac{3}{4}$. The division aspect also lends itself to the idea of mixed fractions, as in 'share 5 pizzas between 3 people', giving $\frac{5}{3}$ or $1\frac{2}{3}$.

Children should be encouraged to recognise a wider range of equivalent fractions. They can be shown that if the numerator and denominator of a fraction have common factors these can be 'cancelled' to give equivalent fractions. Other equivalences could be explored such as:

$$\frac{36}{6} = 6\frac{2}{6} = 6\frac{1}{3}.$$

Children might be asked to:

Find out by cancelling which of these are equivalent fractions. Which is the smallest? Which is the largest?

$$\frac{4}{10}, \frac{16}{40}, \frac{6}{15}, \frac{9}{24}, \frac{2}{5}, \frac{15}{25}.$$

$$\frac{4^2}{10^2} = \frac{2}{5} \quad \frac{16^2}{40^2} = \frac{2}{5} \quad \frac{6^2}{18^2} = \frac{2}{5}$$

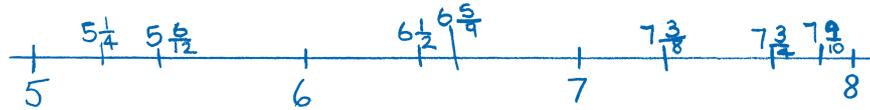
$$\frac{9^3}{24^3} = \frac{3}{8} \quad \frac{15^3}{25^3} = \frac{3}{5}$$

So $\frac{4}{10}, \frac{16}{40}, \frac{6}{15}$ and $\frac{2}{5}$ are equivalent

$\frac{3}{8}$ is the smallest $\frac{3}{5}$ is the largest.

Write down six fractions equivalent to $\frac{3}{4}$.

Put the following fractions on the same number line. $6\frac{1}{2}$, $7\frac{3}{4}$, $5\frac{1}{4}$, $6\frac{5}{8}$, $7\frac{9}{10}$, $5\frac{6}{12}$, $7\frac{3}{8}$.



Decimal equivalents

Once children connect the fraction notation with the operation division, decimal equivalents can easily be found, if necessary using a calculator.

Use a calculator to find the decimal equivalents of $\frac{1}{4}$, $\frac{3}{4}$, $\frac{5}{8}$, $\frac{7}{10}$, $\frac{13}{5}$, $\frac{1}{3}$, $\frac{2}{3}$.

$$\frac{1}{4} = 0.25 \quad \frac{3}{4} = 0.75 \quad \frac{5}{8} = 0.625$$

$$\frac{13}{5} = 2.6 \quad \frac{7}{10} = 0.7$$

$$\frac{1}{3} = 0.3333333 \quad \frac{2}{3} = 0.6666666$$

'I divided 1 by 3 with my calculator and found the answer was 0.3333333. Then I tried 1 divided by 6 and got 0.1666666. But when I tried it on the computer calculator it gave 0.1666666666667. Miss Rahman said that really the sixes go on for ever, but the calculator display only shows some of them. The computer calculator has rounded up the last 6 to make 7.'

Decimal equivalents can be another way of reinforcing the idea of equivalent fractions:

Use a calculator to find the decimal equivalents of these fractions. What do you notice?

$\frac{3}{15}$, $\frac{4}{20}$, $\frac{1}{5}$, $\frac{5}{25}$, $\frac{6}{30}$, $\frac{7}{35}$, $\frac{10}{50}$, $\frac{100}{500}$.

'I noticed that all the answers are the same, 0.2. I think it is because the bottom number is always five times the top number.'

'I looked at $\frac{7}{35}$. If you divide the top number and the bottom number by 7 you get back to $\frac{1}{5}$.'

'I know that $\frac{1}{5}$ equals $\frac{2}{10}$ and that's why it is 0.2.'

Percentages

The idea of a percentage should be introduced as a number of parts out of a hundred – a special subset of fractions. Children will calculate percentages of amounts using mental methods which will depend very much on the numbers involved. The way in which their work is recorded will reflect the method chosen by the children. Small changes in the percentages will suggest different methods, as when calculating 20%, 25% and 35% of £50:

'I found 10% by dividing by 10, that's £5, and then I doubled it.'

$$20\% \text{ of } \pounds 50 = \pounds 10$$

'25% is $\frac{1}{4}$, so I halved £50, that's £25, and then halved again; or I could have divided by 4.'

$$25\% \text{ of } \text{£}50 = \text{£}12.50$$

For 10%, divide by 10, that's £5. Multiply by 3, to find 30% – £15. Then find 5% – half of 10% – that's £2.50. Add them together – £17.50.

$$35\% \text{ of } \text{£}50 = \text{£}17.50$$

Even seemingly difficult calculations such as finding $17\frac{1}{2}\%$ of an amount of money can be carried out by a mental method, with intermediate amounts being written down:

Find $17\frac{1}{2}\%$ of £46

$$10\% \text{ of } \text{£}46 = \text{£}4.60$$

$$5\% \text{ of } \text{£}46 = \text{£}2.30$$

$$2\frac{1}{2}\% \text{ of } \text{£}46 = \text{£}1.15$$

$$\text{So } 17\frac{1}{2}\% \text{ of } \text{£}46 = \text{£}8.05$$

The relation between fractions, decimals and percentages

Fundamental to these methods is the ability to use the connection between fractions, decimals and percentages. Children can fill in those connections they already know on a table and gradually add more:

Fraction	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{7}{10}$	$\frac{9}{10}$	$2\frac{1}{2}$
Decimal	0.5	0.25	0.75	0.6	0.7	0.9	2.5
Percentage	50%	25%	75%	60%	70%	90%	250%

The fraction $\frac{1}{3}$ is awkward as it gives rise to an infinitely recurring decimal 0.3333... and to the percentage $33\frac{1}{3}\%$. Similarly, the fractions $\frac{2}{3}$ and $\frac{1}{7}$ are interesting to consider. But the fraction $\frac{1}{8}$ can be thought of as a half of $\frac{1}{4}$ and so the corresponding decimal is half of 0.25 or 0.125. When the decimal equivalent of $\frac{1}{8}$ is known, those of $\frac{3}{8}$, $\frac{5}{8}$ and $\frac{7}{8}$ are easily calculated by multiplication by 3, 5 or 7 respectively.

Using fractions and percentages to represent data

Children can collect data from 100 people about their favourite sport, TV programme, etc, and use a data handling software package to display the results as a pie chart. They can then label each section as a fraction or a percentage of the whole group. When the idea of percentages is understood, children can work with other sample sizes. If the size of the group is, say, 25, the percentages can be worked out easily. For others they will need to use a calculator.

The most important thing for key stage 2 children is that they have a basic understanding of the concepts of fractions, decimals and percentages and the links between them. More complex calculations with them will come later.

About this publication

Who's it for?

Teachers, mathematics and assessment coordinators and headteachers in primary schools, LEA mathematics advisers, INSET providers and heads of mathematics in secondary schools.

What's it about?

This booklet offers guidance to teachers on teaching written calculation. It builds on the guidance given in *Teaching mental calculation strategies*, and includes a discussion of early informal methods of recording calculations.

Related material

Teaching mental calculation strategies

What's it for?

It lists methods that might be introduced to children and suggests examples for use in the classroom.

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